The Simplex Method

Thomas R. Cameron

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1 Unboundedness

We have seen that the auxiliary method can be used to identify a feasible basic solution, if they exist, or to show that a given LP is infeasible. Another issue that the simplex method can detect is that of an unbounded LP. For example, consider the LP in (1).

$$\begin{aligned} \text{maximize} \quad & z = 3x_1 + 8x_2 \\ \text{subject to} \quad & -5x_1 + 2x_2 \leq 10, \\ & 2x_1 - 3x_2 \leq 6, \\ & x_i \geq 0, \ \forall i \in \{1, 2\} \end{aligned} \tag{1}$$

The initial tableau for the LP in (1) is shown in Table 1. Note that the corresponding basic variables are $\beta^{(0)} = \{3, 4\}$ and non-basic variables $\pi^{(0)} = \{1, 2\}$, with basic solution $\mathbf{x}^{(0)} = [0, 0, 10, 6]$ and $z^{(0)} = 0$.

-5	2	1	0	0	10
2	-3	0	1	0	6
-3	-8	0	0	1	0

Table 1: Initial tableau for the LP in (1).

We select x_1 as the entering variable with pivot entry $a_{2,1} = 2$. Applying row operations results in the tableau shown in Table 2. Note that the corresponding basic variables are $\beta^{(1)} = \{1, 3\}$ and non-basic variables $\pi^{(1)} = \{2, 4\}$, with basic solution $\mathbf{x}^{(1)} = [3, 0, 10, 0]$ and $z^{(1)} = 9$.

0	$-\frac{11}{2}$	1	$\frac{5}{2}$	0	10
2	-3	0	1	0	6
0	$-\frac{25}{2}$	0	$\frac{3}{2}$	1	9

Table 2: Tableau for the LP in (1), after trading x_1 with x_4.

Note that the pivot row of the tableau in Table 2 indicates that we can increase value of z by increasing the value of x_2. However, there are no positive coefficients in the second column of the tableau. Therefore, trading x_2 with any basic variable results in a basic solution that is infeasible, which suggests that no variables place any restriction on the value of x_2; hence, the LP in (1) is unbounded.

2 Cycling

The last issue one might encounter when using the simplex algorithm is that of cycling. Note that there are only a finite number of possible tableau for any given LOP. Hence, if the simplex algorithm does not halt it must be due to cycling. The good news is that in 1977 Robert Bland proved that the use of the least subscript method ensures that the simplex algorithm will not cycle. Before reviewing this proof, we consider a variant of Beale's classical cycling example which demonstrates how the simplex algirthm can cycle when using the most negative method.

Consider the LP in (2).

$$\begin{aligned} \text{maximize} \quad & z = \frac{3}{4}x_1 - 150x_2 + \frac{1}{50}x_3 - 6x_4 \\ \text{subject to} \quad & \frac{1}{4}x_1 - 60x_2 - \frac{1}{25}x_3 + 9x_4 \leq 0, \\ & \frac{1}{2}x_1 - 90x_2 - \frac{1}{50}x_3 + 3x_4 \leq 0, \\ & x_3 \leq 1, \\ & x_i \geq 0, \quad \forall i \in \{1,2,3,4\} \end{aligned} \tag{2}$$

The initial tableau for the LP in (2) is shown in Table 3. Note that the corresponding basic variables are $\beta^{(0)} = \{5,6,7\}$ and non-basic variables $\pi^{(0)} = \{1,2,3,4\}$, with basic solution $\mathbf{x}^{(0)} = [0,0,0,0,0,0,1]$ and $z^{(0)} = 0$.

$\frac{1}{4}$	-60	$-\frac{1}{25}$	9	1	0	0	0	0
$\frac{1}{2}$	-90	$-\frac{1}{50}$	3	0	1	0	0	0
0	0	1	0	0	0	1	0	1
$-\frac{3}{4}$	150	$-\frac{1}{50}$	6	0	0	0	1	0

Table 3: Initial tableau for the LP in (2).

We select x_1 as the entering variable with pivot entry $a_{1,1} = 1/4$. Applying row operations results in the tableau shown in Table 4. Note that the corresponding basic variables are $\beta^{(1)} = \{1,6,7\}$ and non-basic variables $\pi^{(1)} = \{2,3,4,5\}$, with basic solution $\mathbf{x}^{(1)} = [0,0,0,0,0,0,1]$ and $z^{(1)} = 0$.

1	-240	$-\frac{4}{25}$	36	4	0	0	0	0
0	30	$\frac{3}{50}$	-15	-2	1	0	0	0
0	0	1	0	0	0	1	0	1
0	-30	$-\frac{7}{50}$	33	3	0	0	1	0

Table 4: Tableau for the LP in (2), after trading x_1 with x_5.

Next, we select x_2 as the entering variable with pivot entry $a_{2,2} = 30$. Applying row operations results in the tableau shown in Table 5. Note that the corresponding basic variables are $\beta^{(2)} = \{1, 2, 7\}$ and non-basic variables $\pi^{(2)} = \{3, 4, 5, 6\}$, with basic solution $\mathbf{x}^{(2)} = [0, 0, 0, 0, 0, 0, 1]$ and $z^{(2)} = 0$.

1	0	$\frac{8}{25}$	-84	-12	8	0	0	0
0	1	$\frac{1}{500}$	$-\frac{1}{2}$	$-\frac{1}{15}$	$\frac{1}{30}$	0	0	0
0	0	1	0	0	0	1	0	1
0	0	$-\frac{2}{25}$	18	1	1	0	1	0

Table 5: Tableau for the LP in (2), after trading x_2 with x_6.

Next, we select x_3 as the entering variable with pivot entry $a_{1,3} = 8/25$. Applying row operations results in the tableau shown in Table 6. Note that the corresponding basic variables are $\beta^{(3)} = \{2, 3, 7\}$ and non-basic variables $\pi^{(3)} = \{1, 4, 5, 6\}$, with basic solution $\mathbf{x}^{(3)} = [0, 0, 0, 0, 0, 0, 1]$ and $z^{(3)} = 0$.

$\frac{25}{8}$	0	1	$-\frac{525}{2}$	$-\frac{75}{2}$	25	0	0	0
$-\frac{1}{160}$	1	0	$\frac{1}{40}$	$\frac{1}{120}$	$-\frac{1}{60}$	0	0	0
$-\frac{25}{8}$	0	0	$\frac{525}{2}$	$\frac{75}{2}$	-25	1	0	1
$\frac{1}{4}$	0	0	-3	-2	3	0	1	0

Table 6: Tableau for the LP in (2), after trading x_3 with x_1.

Next, we select x_4 as the entering variable with pivot entry $a_{2,4} = \frac{1}{40}$. Applying row operations results in the tableau shown in Table 7. Note that the corresponding basic variables are $\beta^{(4)} = \{3, 4, 7\}$ and non-basic variables $\pi^{(4)} = \{1, 2, 5, 6\}$, with basic solution $\mathbf{x}^{(4)} = [0, 0, 0, 0, 0, 0, 1]$ and $z^{(4)} = 0$.

Next, we select x_5 as the entering variable with pivot entry $a_{1,5} = 50$. Applying row operations results in the tableau shown in Table 8. Note that the corresponding basic variables are $\beta^{(5)} = \{4, 5, 7\}$ and non-basic variables $\pi^{(5)} = \{1, 2, 3, 6\}$, with basic solution $\mathbf{x}^{(5)} = [0, 0, 0, 0, 0, 0, 1]$ and $z^{(5)} = 0$.

Next, we select x_6 as the entering variable with pivot entry $a_{2,6} = 1/3$. Applying row operations results in the tableau shown in Table 9. Note that the corresponding basic variables are $\beta^{(5)} = \{5, 6, 7\}$ and non-basic variables $\pi^{(5)} = \{1, 2, 3, 4\}$, with basic solution

$-\frac{125}{2}$	10500	1	0	50	-150	0	0	0
$-\frac{1}{4}$	40	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0	0	0
$\frac{125}{2}$	-10500	0	0	-50	150	1	0	1
$-\frac{1}{2}$	120	0	0	-1	1	0	1	0

Table 7: Tableau for the LP in (2), after trading x_4 with x_2.

$-\frac{5}{4}$	210	$\frac{1}{50}$	0	1	-3	0	0	0
$\frac{1}{6}$	-30	$-\frac{1}{150}$	1	0	$\frac{1}{3}$	0	0	0
0	0	1	0	0	0	1	0	1
$-\frac{7}{4}$	330	$\frac{1}{50}$	0	0	-2	0	1	0

Table 8: Tableau for the LP in (2), after trading x_5 with x_3.

$\mathbf{x}^{(6)} = [0, 0, 0, 0, 0, 0, 1]$ and $z^{(6)} = 0$.

$\frac{1}{4}$	-60	$-\frac{1}{25}$	9	1	0	0	0	0
$\frac{1}{2}$	-90	$-\frac{1}{50}$	3	0	1	0	0	0
0	0	1	0	0	0	1	0	1
$-\frac{3}{4}$	150	$-\frac{1}{50}$	6	0	0	0	1	0

Table 9: Tableau for the LP in (2), after trading x_6 with x_4.

3 The Fundamental Theorem of Linear Programming

Now, we are ready to state the fundamental theorem of linear optimization.

Theorem 1. *Let P be an LOP in standard form. Then,*

a. P is either infeasible, unbounded, or it has a maximum.

b. If P has a feasible solution, then it has a feasible tableau.

c. If P has an optimal solution, then it has a optimal tableau.