# The Standard Form and the Dual

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## 1 The Standard Form and the Dual

In this section, we introduce basic terminology for linear programs (LPs). As a running example, we will reference the model in (1a)–(1e).

$$\begin{aligned} \text{maximize} \quad & z = x_1 + x_2 && \text{(1a)} \\ \text{subject to} \quad & 3x_1 + 5x_2 \leq 90, && \text{(1b)} \\ & 9x_1 + 5x_2 \leq 180, && \text{(1c)} \\ & x_2 \leq 15, && \text{(1d)} \\ & x_i \geq 0, \ \forall i \in \{1, 2\} && \text{(1e)} \end{aligned}$$

We say that this model is in the standard form since it is a maximization problem, all linear inequalities are less than or equal to a constant, and every variable is non-negative. Later on, we will consider variations on each of these themes; that is, minimization problems, greater than or equal to linear inequalities, and negative variables.

We say that $x_1, x_2$ form a feasible solution to the model in (1a)–(1e) if all inequalities are satisfied. Moreover, a feasible solution is optimal if no other feasible solution has a larger objective value. The set of feasible solutions is shown in Figure 1.

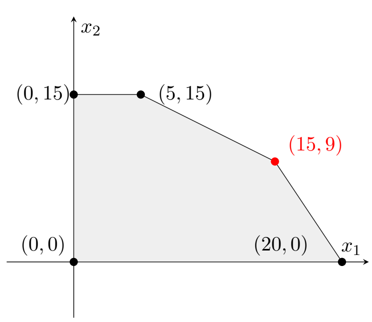


Figure 1: Feasible region for example LP in (1a)–(1e), with optimal solution in red.

Recall from calculus that the extreme points of a function occur at boundary points or at critical points. Since the objective function is linear, it follows that its extreme points occur at the corners of the boundary. Note that the point $(20, 0)$ is a feasible solution and corresponds to an objective value of $z(20, 0) = 20$. Let $z^*$ denote the maximum objective value over the feasible region. Then, we know that $z^* \geq 20$.

To get an upper bound, we use the following analysis

$$\begin{aligned} z = x_1 + x_2 &= \frac{1}{12}\left((3x_1 + 5x_2) + (9x_1 + 5x_2) + 2x_2\right) \\ &\leq \frac{1}{12}\left(90 + 180 + 30\right) = \frac{300}{12} = 25. \end{aligned}$$

Note that we have multiplied each constraint by $y_1 = 1/12$, $y_2 = 1/12$, and $y_3 = 1/6$, respectively. In general, we can obtain an upper bound from the following

$$\begin{aligned} z = x_1 + x_2 &\leq (3y_1 + 9y_2)x_1 + (5y_1 + 5y_2 + y_3)x_2 \\ &= y_1(3x_1 + 5x_2) + y_2(9x_1 + 5x_2) + y_3(x_2) \\ &\leq 90y_1 + 180y_2 + 15y_3 = w. \end{aligned}$$

In order for the first inequality to hold, we require $3y_1 + 9y_2 \geq 1$ and $5y_1 + 5y_2 + y_3 \geq 1$. Under these conditions, we seek non-negative values for $y_1, y_2, y_3$ that minimize the upper bound $w$. That is, we seek a solution to the minimization problem in (2a)–(2d).

$$\begin{aligned} \text{minimize} \quad & w = 90y_1 + 180y_2 + 15y_3 && \text{(2a)} \\ \text{subject to} \quad & 3y_1 + 9y_2 \geq 1, && \text{(2b)} \\ & 5y_1 + 5y_2 + y_3 \geq 1, && \text{(2c)} \\ & y_i \geq 0, \ \forall i \in \{1, 2, 3\} && \text{(2d)} \end{aligned}$$

Note that the LP in (2a)–(2d) is called the dual of the LP in (1a)–(1e), which we refer to as the primal LP. The variables belonging to each problem will also be called primal and dual, respectively. It is important to note that the dual LP has a feasible region that is unbounded. In Figure 2, we plot two facets of the feasible region along with its optimal solution in red.

## 1.1 Class Exercises

I. Find an optimal solution to the model in (1a)–(1e).

II. Show that $y_1 = 2/15$, $y_2 = 1/15$, $y_3 = 0$ is an optimal solution to the dual LP in (2a)–(2d).

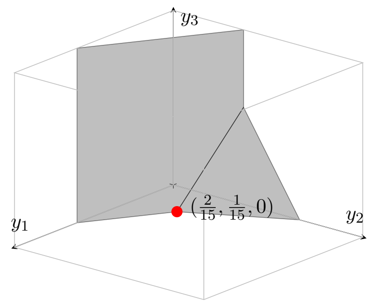


Figure 2: Feasible region for dual LP in (2a)–(2d), with optimal solution in red.

III. Every standard LP can be written in the following matrix form

$$\begin{aligned} \text{maximize} \quad & z = \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Write the dual LP in matrix form and prove the weak duality theorem.