# The Simplex Method

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## 1 Algebraic Viewpoint

Recall the linear program in (1a)–(1e).

$$\begin{aligned} \text{maximize} \quad & z = x_1 + x_2 && \text{(1a)} \\ \text{subject to} \quad & 3x_1 + 5x_2 \leq 90, && \text{(1b)} \\ & 9x_1 + 5x_2 \leq 180, && \text{(1c)} \\ & x_2 \leq 15, && \text{(1d)} \\ & x_i \geq 0, \ \forall i \in \{1, 2\} && \text{(1e)} \end{aligned}$$

By adding slack variables $x_3, x_4, x_5$, we are able to rewrite the inequalities in this model as equalities, see (2).

$$\begin{aligned} \text{maximize} \quad & z = x_1 + x_2 \\ \text{subject to} \quad & x_3 = 90 - 3x_1 - 5x_2, \\ & x_4 = 180 - 9x_1 - 5x_2, \\ & x_5 = 15 - x_2, \\ & x_i \geq 0, \ \forall i \in \{1, 2, 3, 4, 5\} \end{aligned} \qquad (2)$$

We call this formulation a dictionary, and it has the property that the set of variables are split in two. Those on the left side (ignoring the objective variable and non-negativity constraints) are called basic variables, those on the right are non-basic variables. The set of all basic variables is called the basis. We can record the same information in a tableau as shown in Table 1. Note that we have ordered the rows by the constraints with the last row representing the objective function. Also, the columns are ordered by the variables with the last column corresponding to the **b** vector, recall the matrix form of an LP discussed discussed in the lecture on standard forms and duals.

Any set of $x_i$ that satisfy the problem constraints when written as equalities will be referenced as a solution. A feasible solution is one that also satisfies the non-negativity constraints, infeasible solutions do not. A basic solution is one in which all non-basic

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 | 5 | 1 | 0 | 0 | 0 | 90 |
| 9 | 5 | 0 | 1 | 0 | 0 | 180 |
| 0 | 1 | 0 | 0 | 1 | 0 | 15 |
| -1 | -1 | 0 | 0 | 0 | 1 | 0 |

Table 1: Tableau for dictionary in (2).

variables are set to zero. For example, the dictionary in (2) yields the basic solution $\mathbf{x} = [0, 0, 90, 180, 15]$ with $z = 0$. The simplex algorithm evolves by trading basic variables with non-basic variables, which changes the dictionary and corresponding tableau.

We use $\beta$ to denote the set of basic variables and $\pi$ to denote the set of non-basic variables. For example, the dictionary in (2) has basic variables $\beta = \{3, 4, 5\}$ and non-basic variables $\pi = \{1, 2\}$. We call a dictionary (or tableau) feasible if its corresponding basic solution is feasible; otherwise, we say it is infeasible. Whenever we have a tableau that is infeasible, we will say that we are in Phase I of the simplex algorithm; otherwise, we are in Phase II of the simplex algorithm. A dictionary (or tableau) is optimal if its corresponding basic solution is optimal, in this case the simplex algorithm will halt.

From the objective row of the tableau in Table 1, we see that an increase in $x_1$ will increase the value of $z$. For this reason, we identify $x_1$ as the entering variable since it will enter the basis, and we reference column 1 of the tableau in 1 as the pivot column. Now, we must identify a basic variable to trade with. To this end, we find an entry $a_{i,1} > 0$ such that $b_i/a_{i,1}$ forces the tightest restriction on the entering variable $x_1$. In this case, $a_{2,1} = 9$ forces the tightest restriction. Note that the corresponding constraint is $x_4 = 180 - 9x_1 - 5x_2$; hence, $x_1$ is trading with the basic variable $x_4$. Finally, we treat $a_{2,1}$ as the pivot entry of the pivot column and we apply row operations to make all other entries in that column zero. The resulting tableau is shown in Table 2 and the corresponding dictionary is shown in (3). Note that this dictionary has basic variables $\beta^{(1)} = \{1, 3, 5\}$ and non-basic variables $\pi^{(1)} = \{2, 4\}$. Hence, the basic solution is given by $\mathbf{x} = [20, 0, 30, 0, 15]$ with $z = 20$.

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | $\frac{10}{3}$ | 1 | $-\frac{1}{3}$ | 0 | 0 | 30 |
| 9 | 5 | 0 | 1 | 0 | 0 | 180 |
| 0 | 1 | 0 | 0 | 1 | 0 | 15 |
| 0 | $-\frac{4}{9}$ | 0 | $\frac{1}{9}$ | 0 | 1 | 20 |

Table 2: Tableau for dictionary in (3).

$$
\begin{aligned}
\text{maximize} \quad & z = 20 + \frac{4}{9}x_2 - \frac{1}{9}x_4 \\
\text{subject to} \quad & x_3 = 30 - \frac{10}{3}x_2 + \frac{1}{3}x_4, \\
& 9x_1 = 180 - 5x_2 - x_4, \\
& x_5 = 15 - x_2, \\
& x_i \geq 0,\ \forall i \in \{1,2,3,4,5\}
\end{aligned}
\tag{3}
$$

Before proceeding, note that by selecting the pivot entry $a_{i,j}$ of the pivot column $j$ so that $b_i/a_{i,j}$ forces the tightest restriction on the entering variable $x_j$ ensures that the resulting tableau will be feasible, so we remain in Phase II of the Simplex Algorithm. As an example, consider what would have happened to the tableau in Table 1 if we selected $a_{1,1} = 3$ as the pivot entry.

From the objective row of the tableau in Table 2, we see that an increase in $x_2$ will increase the value of $z$. Hence, $x_2$ is the entering variable and column 2 of the tableau in Table 2 is the pivot column. Furthermore, the entry $a_{1,2} = \frac{10}{3}$ forces the tightest restriction on the entering variable $x_2$. Note that the corresponding constraint is $x_3 = 30 - \frac{10}{3}x_2 + \frac{1}{3}x_4$; hence, $x_2$ is trading with the basic variable $x_3$. Moreover, we select $a_{1,2}$ as the pivot entry of the pivot column and we apply row operations to make all other entries in that column zero. The resulting tableau is shown in Table 3 and the corresponding dictionary is shown in (4). Note that this dictionary has basic variables $\beta^{(2)} = \{1,2,5\}$ and non-basic variables $\pi^{(2)} = \{3,4\}$. Hence, the basic solution is given by $\mathbf{x} = [15, 9, 0, 0, 6]$ with $z = 24$. At this pint, the value of $z$ cannot be increased so the Simplex Algorithm halts.

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | $\frac{10}{3}$ | 1 | $-\frac{1}{3}$ | 0 | 0 | 30 |
| 9 | 0 | $-\frac{3}{2}$ | $\frac{3}{2}$ | 0 | 0 | 135 |
| 0 | 0 | $-\frac{3}{10}$ | $\frac{1}{10}$ | 1 | 0 | 6 |
| 0 | 0 | $\frac{2}{15}$ | $\frac{1}{15}$ | 0 | 1 | 24 |

Table 3: Tableau for dictionary in (4).

$$
\begin{aligned}
\text{maximize} \quad & z = 24 - \frac{2}{15}x_3 - \frac{1}{15}x_4 \\
\text{subject to} \quad & \frac{10}{3}x_2 = 30 - x_3 + \frac{1}{3}x_4, \\
& 9x_1 = 135 + \frac{3}{2}x_3 - \frac{3}{2}x_4, \\
& x_5 = 6 + \frac{3}{10}x_3 - \frac{1}{10}x_4, \\
& x_i \geq 0,\ \forall i \in \{1,2,3,4,5\}
\end{aligned}
\tag{4}
$$

Finally, we note that there may be several choices for the pivot column on a particular iteration of the Simplex Algorithm. We have always chosen the column with the least subscript; this is the Least Subscript Method. Another choice would be the column corresponding to the most negative coefficient in the objective row, which would force $z$ to increase at the highest rate; this is the Most Negative Method. A third choice would be to pick the column that actually corresponds to the largest increase in $z$; this is the Greatest Increase Method. For simplicity, we will always use the Least Subscript Method. Moreover, once a pivot column has been selected, there may be multiple choices for a pivot that forces the tightest restriction on the entering variable. In this case, we will also select the pivot entry with the least subscript.