Linear Programming

Thomas R. Cameron

January 14, 2026

1 Linear Programming

A *linear program (LP)* is an optimization problem where the objective function and constraints are linear. The standard form of a linear program is

$$\text{maximize} \quad z = \mathbf{c}^T\mathbf{x} \tag{1a}$$

$$\text{subject to} \quad A\mathbf{x} \leq \mathbf{b} \tag{1b}$$

1.1 The Diet Problem

Imagine your entire class must eat from the menu in Table 1. Each meal must satisfy certain requirements on the percentage of vitamin intake and the goal is to minimize the total number of calories.

	$\%A$	$\%C$	$\%Calc$	$\%Iron$	$Calories$
Hamburger	4	4	10	15	250
Chicken	8	15	15	8	400
Fish	2	0	15	10	370
Cheeseburger	15	6	30	20	490
Requirements	10	10	15	15	

Table 1: Class menu for diet problem.

Let y_1, y_2, y_3, y_4 denote non-negative continuous variables which represent how much hamburger, chicken, fish, or cheeseburger is consumed, respectively. Then, the diet problem model is shown in (2a)–(2f)

$$\begin{aligned}
\text{minimize} \quad & w = 250y_1 + 400y_2 + 370y_3 + 490y_4 && \text{(2a)}\\
\text{subject to} \quad & 4y_1 + 8y_2 + 2y_3 + 15y_4 \geq 10, && \text{(2b)}\\
& 4y_1 + 15y_2 + 0y_3 + 6y_4 \geq 10, && \text{(2c)}\\
& 10y_1 + 15y_2 + 15y_3 + 30y_4 \geq 15, && \text{(2d)}\\
& 15y_1 + 8y_2 + 10y_3 + 20y_4 \geq 15, && \text{(2e)}\\
& y_i \geq 0,\ \forall i \in \{1,2,3,4\} && \text{(2f)}
\end{aligned}$$

Note that the objective function of the diet problem, denoted w, is attained from the Calories column of Table 1. Furthermore, each constraint of the diet problem follow from the nutritional requirements. Since the objective function is linear and each constraint is a linear inequality, the diet problem is an example of a linear program.

We say that y_1, y_2, y_3, y_4 form a feasible solution to the diet problem if constraints (2b)–(2f) are satisfied. Moreover, a feasible solution is optimal if it is minimal with respect to the objective function in (2a), that is, no other feasible solution corresponds to a smaller objective function value.

We can obtain lower bounds on the objective function for feasible solutions by manipulating the constraints of the diet problem. For example, if we multiply the constraint (2d) by 15 we obtain $150y_1 + 225y_2 + 225y_3 + 450y_4 \geq 225$, which leads to the following lower bound

$$\begin{aligned}
w &= 250y_1 + 400y_2 + 370y_3 + 490y_4\\
&\geq 150y_1 + 225y_2 + 225y_3 + 450y_4 \geq 225.
\end{aligned}$$

We can improve our lower bound to 350 by multiply constraints (2b)–(2c) and (2e) by 10 and then adding them together. From the exercises in 1.2, we know that the optimal value of the objective function lies somewhere between 400 and 500. While considering lower bounds on the diet problem, we were in the process of constructing its dual LP shown in (3a)–(3f).

$$\begin{aligned}
\text{maximize} \quad & z = 10x_1 + 10x_2 + 15x_3 + 15x_4 && \text{(3a)}\\
\text{subject to} \quad & 4x_1 + 4x_2 + 10x_3 + 15x_4 \leq 250, && \text{(3b)}\\
& 8x_1 + 15x_2 + 15x_3 + 8x_4 \leq 400, && \text{(3c)}\\
& 2x_1 + 0x_2 + 15x_3 + 10x_4 \leq 370, && \text{(3d)}\\
& 15x_1 + 6x_2 + 30x_3 + 20x_4 \leq 490, && \text{(3e)}\\
& x_i \geq 0,\ \forall i \in \{1,2,3,4\} && \text{(3f)}
\end{aligned}$$

1.2 Class Exercises

I. Find a feasible solution to the diet problem that consumes at most 500 calories.

II. Find a lower bound for the diet problem of at least 400.

III. Let y_1, y_2, y_3, y_4 denote a feasible solution to the diet problem with corresponding objective value w. Also, let x_1, x_2, x_3, x_4 denote a feasible solution to the dual problem with corresponding objective value z. Show that $z \leq w$.