

# The Lagrangian

Thomas R. Cameron

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## 1 Introduction

Consider the QP in the following form

$$\begin{aligned} & \text{minimize} && \frac{1}{2} \mathbf{x}^T Q \mathbf{x} + \mathbf{c}^T \mathbf{x} \\ & \text{subject to} && A \mathbf{x} = \mathbf{b} \end{aligned}$$

where  $Q \in \mathbb{R}^{n \times n}$  is symmetric,  $A \in \mathbb{R}^{m \times n}$ ,  $\mathbf{b} \in \mathbb{R}^m$ , and  $\mathbf{c} \in \mathbb{R}^n$ . In the previous lecture, we showed that at an optimal solution  $\mathbf{x}^*$  satisfies

$$Q \mathbf{x}^* + \mathbf{c} \perp \text{nul}(A),$$

which implies

$$Q \mathbf{x}^* + \mathbf{c} \in \text{row}(A) = \text{col}(A^T) = \text{col}(-A^T)$$

Therefore, there exists  $\mathbf{y}$  such that

$$Q \mathbf{x}^* + \mathbf{c} = -A^T \mathbf{y}.$$

This is the first-order optimality condition. When combined with the feasibility condition  $A \mathbf{x} = \mathbf{b}$ , we get the KKT system which is written as

$$\begin{aligned} Q \mathbf{x} + \mathbf{c} + A^T \mathbf{y} &= 0 \\ A \mathbf{x} - \mathbf{b} &= 0 \end{aligned}$$

In this lecture, we introduce the Lagrangian function which encodes the KKT system as its critical points. The Lagrangian will be useful in extending our framework to include inequality constraints, non-linear constraints, and duality theory. Moreover, the Lagrangian also encodes the second-order optimality conditions which when combined with the first-order conditions provide necessary and sufficient conditions for optimality.

## 2 The Lagrangian

Let  $f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^T Q \mathbf{x} + \mathbf{c}^T \mathbf{x}$  denote the quadratic in the QP. The lagrangian function is defined by

$$L(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}) + \mathbf{y}^T (A \mathbf{x} - \mathbf{b}).$$

Note that if  $A \mathbf{x} = \mathbf{b}$ , then  $L(\mathbf{x}, \mathbf{y}) = f(\mathbf{x})$ . Otherwise, the term  $\mathbf{y}^T (A \mathbf{x} - \mathbf{b})$  penalizes infeasibility. In this case, the  $\mathbf{y}$  controls how strong constraint violations are penalized.

## 2.1 Recovering the KKT System

We compute the gradients of the Lagrangian.

$$\nabla_{\mathbf{x}}L = Q\mathbf{x} + \mathbf{c} + A^T\mathbf{y}$$

$$\nabla_{\mathbf{y}}L = A\mathbf{x} - \mathbf{b}$$

Setting both equal to zero gives:

$$Q\mathbf{x} + \mathbf{c} = A^T\mathbf{y}, \quad A\mathbf{x} = \mathbf{b}.$$

Therefore, the KKT system is exactly the system of equations obtained by finding a critical point of the Lagrangian.

## 2.2 Existence and uniqueness

In the previous lecture we proved the following result

**Theorem 2.1.** *Suppose that  $Q$  is positive definite on  $\text{nul}(A)$ , that is,  $\mathbf{d}^T Q \mathbf{d} > 0$  for all non-zero  $\mathbf{d} \in \text{nul}(A)$ . If  $A\mathbf{x} = \mathbf{b}$  is feasible, then the QP has a unique minimizer.*

This theorem provides sufficient conditions for a unique optimal solution of the QP. Just as the KKT system, the Lagrangian encodes these conditions. Indeed, note that

$$\nabla_{xx}^2 L(\mathbf{x}, \mathbf{y}) = Q.$$

Therefore, a feasible QP has a unique minimizer if

$$\mathbf{d}^T \nabla_{xx}^2 L(\mathbf{x}, \mathbf{y}) \mathbf{d} > 0$$

for all non-zero  $\mathbf{d} \in \text{nul}(A)$ .

## 3 Example

Consider the QP with

$$Q = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}.$$

The Lagrangian

$$\begin{aligned} L(\mathbf{x}, \mathbf{y}) &= \frac{1}{2} \mathbf{x}^T Q \mathbf{x} + \mathbf{c}^T \mathbf{x} + \mathbf{y}^T (A\mathbf{x} - \mathbf{b}) \\ &= \frac{1}{2} (x_1^2 - 2x_1x_2 + 2x_2^2 - 2x_2x_3 + x_3^2) - 2x_1 + y_1(x_2 - 1) + y_2(x_1 + x_3) \end{aligned}$$

is a quadratic that encodes both the objective and constraints of the QP.

### 3.1 Critical Points

The critical points of the Lagrangian are given by the KKT system

$$\begin{aligned}\nabla_{\mathbf{x}}L(\mathbf{x}, \mathbf{y}) &= Q\mathbf{x} + \mathbf{c} + A^T\mathbf{y} = 0, \\ \nabla_{\mathbf{y}}L(\mathbf{x}, \mathbf{y}) &= A\mathbf{x} - \mathbf{b} = 0.\end{aligned}$$

In this particular example, we have

$$\begin{aligned}x_1 - x_2 - 2 + y_2 &= 0, \\ -x_1 + 2x_2 - x_3 + y_1 &= 0, \\ -x_2 + x_3 + y_2 &= 0, \\ x_2 - 1 &= 0, \\ x_1 + x_3 &= 0.\end{aligned}$$

Note that  $x_2 = 1$ ,  $x_3 = -x_1$ , and

$$x_1 - x_2 - 2 = -x_2 + x_3.$$

Therefore,

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$

### 3.2 Hessian

The Hessian (in  $\mathbf{x}$ ) of the Lagrangian is given by

$$\nabla_{xx}^2L(\mathbf{x}, \mathbf{y}) = Q.$$

The second-order optimality conditions state that the critical point of the Lagrangian is the unique minimizer of the LOP if  $Q$  is positive definite on  $\text{nul}(A)$ . Note that

$$\text{nul}(A) = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right).$$

Therefore, every non-zero  $\mathbf{d} \in \text{nul}(A)$  is of the form

$$\mathbf{d} = t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Furthermore,  $\mathbf{d}^T Q \mathbf{d} = 2t^2 > 0$  for all  $t \neq 0$ . Hence,  $Q$  is positive definite on  $\text{nul}(A)$ . So, the KKT solution corresponds to the unique minimizer of the QP.