# The Simplex Method

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## 1 Infeasibility

When the given LP has an initial dictionary (or tableau) that is infeasible, we use the auxiliary method to move the simplex algorithm from Phase I to Phase II. In general, given a LP in standard form

$$\begin{aligned} \text{maximize} \quad & z = \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{i,j} x_j \leq b_i,\ 1 \leq i \leq m, \\ & x_j \geq 0,\ 1 \leq j \leq n \end{aligned}$$

the auxiliary problem is defined as follows

$$\begin{aligned} \text{maximize} \quad & v = -x_0 \\ \text{subject to} \quad & -x_0 + \sum_{j=1}^{n} a_{i,j} x_j \leq b_i,\ 1 \leq i \leq m, \\ & x_j \geq 0,\ 0 \leq j \leq n \end{aligned}$$

Theorem 1 describes an important relationship between the primal LP and the auxiliary LP.

**Theorem 1.** *Let $P$ denote a primal LP in standard form and let $Q$ denote the corresponding auxiliary LP. Then, $P$ is feasible if and only if $Q$ is optimal at $v^* = 0$.*

*Proof.* Suppose $P$ is feasible and let $\mathbf{x} = [x_1, \ldots, x_n]$ denote a feasible solution of $P$. Then,

$$\sum_{j=1}^{n} a_{i,j} x_j \leq b_i,$$

for $1 \leq i \leq m$. Therefore, every constraint of $Q$ is satisfied for $x_0 = 0$. Since the objective function of the auxiliary problem is to maximize $v = -x_0$, where $x_0 \geq 0$, it follows that this solution is $Q$ optimal and $v^* = 0$.

Conversely, suppose that $Q$ is optimal at $v^* = 0$. Then, $x_0 = 0$, which implies that every constraint of $Q$ is of the form

$$\sum_{j=1}^{n} a_{i,j} x_j \leq b_i,$$

for $1 \leq i \leq m$. Therefore $\mathbf{x} = [x_1, \ldots, x_n]$ is a feasible solution of $P$. □

From the proof of Theorem 1, we see that every $Q$-optimal solution at $v^* = 0$ can be used to construct a $P$-feasible solution. Moreover, Theorem 2 shows how to construct a feasible tableau for the auxiliary LOP.

**Theorem 2.** *Suppose that $x_j$ is a (basic) slack variable for a given auxiliary LP with an infeasible tableau. Then, trading $x_0$ with $x_j$ yields a feasible dictionary (or tableau) if and only if the value of $x_j$ in the basic solution is most negative.*

*Proof.* Without loss of generality, we can assume that the coefficients for each slack variable in a tableau are 1. Therefore, a tableau is infeasible if and only if there exists a negative $b_j$, for some $1 \leq j \leq m$. When trading $x_0$ with $x_j$, where $n+1 \leq j \leq n+m$, we perform the following row operations

$$-b_{j-n} + b_k,$$

for all $1 \leq k \leq n$, where $k \neq j - n$. So, trading $x_0$ with $x_j$ results in a feasible tableau if and only if $b_{j-n}$ is the most negative value, that is, the value of $x_j$ in the basic solution is most negative. □

For example, consider the following LP whose initial tableau is shown in Table 1.

$$\begin{aligned} \text{maximize} \quad & z = -90x_1 - 180x_2 - 15x_3 \\ \text{subject to} \quad & -3x_1 - 9x_2 \leq -1, \\ & -5x_1 - 5x_2 - x_3 \leq -1, \\ & x_i \geq 0, \ \forall i \in \{1, 2, 3\} \end{aligned} \tag{1}$$

| | | | | | | |
|---|---|---|---|---|---|---|
| -3 | -9 | 0 | 1 | 0 | 0 | -1 |
| -5 | -5 | -1 | 0 | 1 | 0 | -1 |
| 90 | 180 | 15 | 0 | 0 | 1 | 0 |

Table 1: Initial Tableau for LP in (1).

Note that the basic solution to the tableau in Table 1 is $\mathbf{x} = [0, 0, 0, -1, -1]$, which is not feasible. Hence, we are in Phase I of the Simplex Algorithm. The auxiliary LP is shown below with initial tableau shown in Table 2.

$$\begin{aligned} \text{maximize} \quad & v = -x_0 \\ \text{subject to} \quad & -x_0 - 3x_1 - 9x_2 \leq -1, \\ & -x_0 - 5x_1 - 5x_2 - x_3 \leq -1, \\ & x_i \geq 0, \ \forall i \in \{0, 1, 2, 3\} \end{aligned} \tag{2}$$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| -1 | -3 | -9 | 0 | 1 | 0 | 0 | -1 |
| -1 | -5 | -5 | -1 | 0 | 1 | 0 | -1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

Table 2: Initial Tableau for Auxiliary LP in (2).

Following Theorem 2, we select $x_0$ as the entering variable and column 0 as the pivot column, with pivot entry $a_{0,0} = -1$. Note that $x_0$ is trading with $x_4$. Now, we apply row operations to make all other entries in the pivot column zero. The resulting tableau is shown in Table 3.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 3 | 9 | 0 | -1 | 0 | 0 | 1 |
| 0 | -2 | 4 | -1 | -1 | 1 | 0 | 0 |
| 0 | -3 | -9 | 0 | 1 | 0 | 1 | -1 |

Table 3: Auxiliary tableau after trading $x_0$ with $x_4$.

Note that the tableau in Table 3 corresponds to basic variables $\beta^{(1)} = \{0, 5\}$ and non-basic variables $\pi^{(1)} = \{1, 2, 3, 4\}$. Hence, the basic solution is $\mathbf{x}^{(1)} = [1, 0, 0, 0, 0, 0]$, which is feasible. Therefore, we are now in Phase II of the simplex algorithm. Next, we select $x_1$ as the pivot column, with pivot entry $a_{0,1} = 3$. Note that $x_1$ is trading with $x_0$. Applying row operations gives us the tableau shown in Table 4.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 3 | 9 | 0 | -1 | 0 | 0 | 1 |
| $\frac{2}{3}$ | 0 | 10 | -1 | $-\frac{5}{3}$ | 1 | 0 | $\frac{2}{3}$ |
| 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

Table 4: Auxiliary tableau after trading $x_1$ with $x_0$.

Note that the tableau in Table 4 corresponds to basic variables $\beta^{(2)} = \{1, 5\}$ and non-basic variables $\pi^{(2)} = \{0, 2, 3, 4\}$. Hence, the basic solution is $\mathbf{x}^{(2)} = [0, \frac{1}{3}, 0, 0, 0, \frac{2}{3}]$. Moreover, this tableau is optimal with corresponding value $v^* = 0$. So, Theorem 1 implies that the primal LP in (1) is feasible. To recover a corresponding feasible tableau, we use the tableau in Table 4 by dropping the variable $x_0$, which has a value of 0. Moreover, since the objective function should not include any basic variables, we restate the objective variable in terms of the variables $x_2, x_3, x_4$.

$$\begin{aligned} z &= -90x_1 - 180x_2 - 15x_3 \\ &= -30 \cdot 3x_1 - 180x_2 - 15x_3 \\ &= -30\,(1 - 9x_2 + x_4) - 180x_2 - 15x_3 \\ &= -30 + 90x_2 - 15x_3 - 30x_4 \end{aligned}$$

Hence, we have the following feasible tableau

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 | 9 | 0 | -1 | 0 | 0 | 1 |
| 0 | 10 | -1 | $-\frac{5}{3}$ | 1 | 0 | $\frac{2}{3}$ |
| 0 | -90 | 15 | 30 | 0 | 1 | -30 |

Table 5: Feasible tableau for LP in (1).

Now that we are in Phase II of the simplex algorithm, we can proceed to identify the optimal tableau. To this end, let $x_2$ be the entering variable and column 2 the pivot column, with pivot entry $a_{2,2} = 10$. Then, applying row operations gives us

| | | | | | | |
|---|---|---|---|---|---|---|
| 3 | 0 | $\frac{9}{10}$ | $\frac{1}{2}$ | $-\frac{9}{10}$ | 0 | $\frac{2}{5}$ |
| 0 | 10 | -1 | $-\frac{5}{3}$ | 1 | 0 | $\frac{2}{3}$ |
| 0 | 0 | 6 | 15 | 9 | 1 | -24 |

Table 6: Optimal tableau for LP in (1).