Inequality-Constrained Quadratic Programming

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1 Introduction

Previously, we studied unconstrained quadratic programs and equality-constrained quadratic programs. In the unconstrained setting, optimal solutions occur at critical points of the quadratic. In the equality-constrained setting, optimal solutions occur at points where the gradient is orthogonal to all feasible directions, which led to the KKT system. The Lagrangian encoded the KKT system that lead to the min-max dual problem and weak and strong duality.

We now extend this framework to inequality-constrained quadratic programs. The key new feature is that the set of feasible directions depends on the point under consideration. In particular, some constraints may be active and behave like equalities, while others are inactive and play no role locally. The central challenge is that we do not know in advance which constraints will be active at an optimal solution.

2 Inequality-Constrained QPs

We consider quadratic programs of the form

$$\begin{aligned} &\text{minimize} \quad && \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \quad && A\mathbf{x} \leq \mathbf{b} \end{aligned}$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric, $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$.

The feasible region is given by

$$\mathcal{F} = \left\{\mathbf{x} \in \mathbb{R}^n \colon A\mathbf{x} \leq \mathbf{b}\right\}.$$

Geometrically, $\mathcal{F}$ is a polyhedron defined by the intersection of half-spaces. Let $\mathbf{x} \in \mathcal{F}$. We say that the ith constraint is active at $\mathbf{x}$ if

$$\mathbf{a}_i^T\mathbf{x} = b_i,$$

and inactive if $\mathbf{a}_i^T\mathbf{x} < b_i$. At an interior point, no constraints are active and the problem behaves locally like an unconstrained quadratic program. At a boundary point, the active constraints determine the local geometry of the feasible region. We denote by $\mathcal{A}(\mathbf{x})$ the set of indices corresponding to active constraints at $\mathbf{x}$.

2.1 Geometry of the Feasible Region

In the equality-constrained case, feasible directions were precisely the vectors in $\text{nul}(A)$. In the inequality-constrained case, the situation is more subtle. Let $\mathbf{x} \in \mathcal{F}$. A vector $\mathbf{d} \in \mathbb{R}^n$ is called a feasible direction at $\mathbf{x}$ if there exists $\epsilon > 0$ such that

$$\mathbf{x} + t\mathbf{d} \in \mathcal{F},$$

for all $t \in [0, \epsilon]$. The following theorem characterizes feasible directions.

Theorem 2.1. *Let $\mathbf{x} \in \mathcal{F}$. Then, $\mathbf{d}$ is a feasible direction at $\mathbf{x}$ if and only if*

$$\mathbf{a}_i^T \mathbf{d} \leq 0,$$

for all $i \in \mathcal{A}(\mathbf{x})$.

Proof. Suppose $\mathbf{d}$ is a feasible direction. Then, there is an $\epsilon > 0$ such that $\mathbf{x} + t\mathbf{d} \in \mathcal{F}$ for all $t \in [0, \epsilon]$. Hence, for each $i \in \mathcal{A}(\mathbf{x})$,

$$\mathbf{a}_i^T (\mathbf{x} + t\mathbf{d}) \leq b_i,$$

for all $t \in [0, \epsilon]$. Since $\mathbf{a}_i^T \mathbf{x} = b_i$, we obtain

$$t\mathbf{a}_i^T \mathbf{d} \leq 0,$$

for all $t \in [0, \epsilon]$. Thus, $\mathbf{a}_i^T \mathbf{d} \leq 0$.

Conversely, suppose that $\mathbf{a}_i^T \mathbf{d} \leq 0$ for all $i \in \mathcal{A}(\mathbf{x})$. Then, for each $i \in \mathcal{A}(\mathbf{x})$, we have

$$\mathbf{a}_i^T (\mathbf{x} + t\mathbf{d}) \leq \mathbf{a}_i^T \mathbf{d} = b_i.$$

For $i \notin \mathcal{A}(\mathbf{x})$, $\mathbf{a}_i^T \mathbf{x} < b_i$. Hence, there exists an $\epsilon > 0$ such that

$$\mathbf{a}_i^T (\mathbf{x} + t\mathbf{d}) \leq b_i,$$

for all $t \in [0, \epsilon]$. Thus, $\mathbf{d}$ is a feasible direction at $\mathbf{x}$. □

Thus, feasible directions are determined entirely by the active constraints. In contrast to the equality-constrained case, these directions form a convex cone rather than a subspace. A convex cone is a subset $\mathcal{C} \subseteq \mathbb{R}^n$ such that for each $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{C}$, $\alpha\mathbf{d}_1 + \beta\mathbf{d}_2 \in \mathcal{C}$ for all $\alpha, \beta \geq 0$. Given a feasible $\mathbf{x} \in \mathcal{F}$, we define the feasible direction cone by

$$\mathcal{C}_{\mathbf{x}} = \left\{ \mathbf{d} \in \mathbb{R}^n \colon \mathbf{a}_i^T \mathbf{d} \leq 0, \quad \forall i \in \mathcal{A}(\mathbf{x}) \right\}.$$

Example

Consider the following QP:

$$\begin{aligned} \text{minimize} \quad & z = x_1^2 + x_1 x_2 + 2x_2^2 - 4x_1 - 6x_2 \\ \text{subject to} \quad & x_1 + x_2 \leq 2, \\ & x_2 \leq \frac{3}{2}, \\ & x_1, x_2 \geq 0 \end{aligned}$$

The feasible region is shown in Figure 1. At the point $\mathbf{x} = (1,1)$, the constraint $x_1 + x_2 \leq 2$ is active. We will reference this constraint by $\mathbf{a}_1^T = (1,1)$. Moreover, at $\mathbf{x}$, the direction $\mathbf{d} = (1,1)$ is not feasible since $\mathbf{a}_1^T\mathbf{d} > 0$. In contrast, the direction $\mathbf{d} = (-1/2, 1/2)$ is feasible since $\mathbf{a}_1^T\mathbf{d} = 0$. The feasible directions at $\mathbf{x}$ form the cone

$$\mathcal{C}_{\mathbf{x}} = \left\{ \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} : d_1 + d_2 \leq 0 \right\}.$$

□

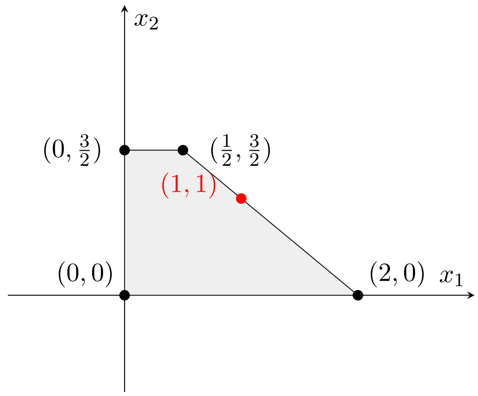

Figure 1: Feasible region of QP with optimal point in red.

2.2 Optimality via Feasible Directions

We now extend the first-order optimality condition to inequality constrained QPs.

Theorem 2.2. *Let $\mathbf{x}^*$ be an optimal solution. Then, for every feasible direction $\mathbf{d} \in \mathcal{C}_{\mathbf{x}^*}$,*

$$\nabla f(\mathbf{x}^*)^T \mathbf{d} \geq 0.$$

Proof. Note that the directional derivative of $f(\mathbf{x})$ at $\mathbf{x}^*$ in the direction $\mathbf{d}$ is given by

$$f'(\mathbf{x}^*; \mathbf{d}) = \nabla f(\mathbf{x}^*)^T \mathbf{d}.$$

For sufficiently small t,

$$\begin{aligned} f(\mathbf{x}^* + t\mathbf{d}) &\approx f(\mathbf{x}^*) + t f'(\mathbf{x}^*; \mathbf{d}) \\ &= f(\mathbf{x}^*) + t\nabla f(\mathbf{x}^*)^T \mathbf{d}. \end{aligned}$$

If there exists a feasible direction $\mathbf{d}$ such that $\nabla f(\mathbf{x}^*)^T \mathbf{d} < 0$, then for sufficiently small $t > 0$,

$$f(\mathbf{x}^* + t\mathbf{d}) < f(\mathbf{x}^*),$$

which contradicts optimality. □

Example

Consider the QP from Figure 1. Note that the gradient is given by

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x_1 + x_2 - 4 \\ 4x_2 + x_1 - 6 \end{bmatrix}.$$

The point $\mathbf{x}^* = (1, 1)$ is optimal. Note that

$$\nabla f(\mathbf{x}^*) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}.$$

Recall that the feasible direction cone is

$$\mathcal{C}_{\mathbf{x}^*} = \left\{ \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} : d_1 + d_2 \leq 0 \right\}.$$

The first-order optimality conditions state that a necessary condition of optimality is that no feasible direction decreases the objective. Indeed, for any $\mathbf{d} \in \mathcal{C}_{\mathbf{x}^*}$, we have

$$\nabla f(\mathbf{x}^*)^T \mathbf{d} = -d_1 - d_2 \geq 0.$$

The point $\mathbf{z} = (2, 0)$ is not optimal. Note that $\mathbf{d} = (-1, 1)$ is a feasible direction at $(2, 0)$. However,

$$\begin{aligned} \nabla f(\mathbf{z})^T \mathbf{d} &= \begin{bmatrix} 0 & -4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= 0 - 4 = -4. \end{aligned}$$

Hence, $\mathbf{z}$ does not satisfy the necessary conditions of optimality. □