Math 482
Homework: QP Geometry, KKT, and Duality

Relevant Topics: equality-constrained QP, feasible directions, KKT conditions, Lagrangian, duality, inequality-constrained QP, active set method

Instructions. Show enough work to justify your answers. You may use online computational tools (such as Wolfram Alpha) to assist with algebraic computations (e.g., solving linear systems), but you must clearly present the key steps and reasoning in your solution. Answers without justification will not receive full credit.

1. **Equality-Constrained QP and Feasible Directions**

   Learning objective: Understand optimization over the null space and the geometric meaning of first-order optimality.

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \end{aligned}$$

   where

$$Q = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} -3 \\ 0 \\ -3 \end{bmatrix},$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

   (a) Find a particular solution $\mathbf{x}_p$ to $A\mathbf{x} = \mathbf{b}$ and a basis for nul(A).

   (b) Write every feasible point as $\mathbf{x} = \mathbf{x}_p + t\mathbf{d}$, where $\mathbf{d} \in \text{nul}(A)$.

   (c) Reduce the problem to a quadratic in t and find the optimal solution $\mathbf{x}^*$.

   (d) Compute $\mathbf{d}^T Q\mathbf{d}$ and explain why Q is positive semidefinite but positive definite on nul(A).

   (e) Show that $Q\mathbf{x}^* + \mathbf{c}$ is orthogonal to all feasible directions and interpret this geometrically.

2. **KKT System and the Lagrangian**

   Learning objective: Understand how the Lagrangian encodes optimality and why second-order conditions are restricted to feasible directions.

   Consider the same QP as in Problem 1.

   (a) Write the Lagrangian $L(\mathbf{x}, \mathbf{y})$.

(b) Compute $\nabla_{\mathbf{x}} L$ and $\nabla_{\mathbf{y}} L$, and write the KKT system.

(c) Solve the KKT system and verify agreement with Problem 1.

(d) Compute $\nabla^2_{\mathbf{xx}} L$ and explain why second-order conditions are checked on $\mathrm{nul}(A)$.

(e) Verify $\mathbf{d}^T Q \mathbf{d} > 0$ for all nonzero $\mathbf{d} \in \mathrm{nul}(A)$ and explain why this implies a unique minimizer.

3. **Duality via Completing the Square**

Learning objective: Derive the dual problem and interpret strong duality.

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 4x_2 \\ \text{subject to} \quad & x_1 + x_2 = 2 \end{aligned}$$

(a) Write the Lagrangian $L(\mathbf{x}, \mathbf{y})$.

(b) Minimize $L(\mathbf{x}, \mathbf{y})$ over $\mathbf{x}$ by completing the square.

(c) Derive the dual function $g(\mathbf{y})$.

(d) Solve the dual problem and recover $\mathbf{x}^*$.

(e) Compute $g(\mathbf{y}^*)$ and $f(\mathbf{x}^*)$, and explain weak and strong duality in this example.

4. **Inequality-Constrained Geometry and KKT**

Learning objective: Connect feasible direction cones, multipliers, and KKT conditions.

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = x_1^2 + x_2^2 - 2x_1 - 2x_2 \\ \text{subject to} \quad & x_1 + x_2 \le 2, \\ & x_1 \ge 0, \\ & x_2 \ge 0 \end{aligned}$$

Let $\mathbf{x}^* = (1, 1)$.

(a) Sketch the feasible region and identify active constraints at $\mathbf{x}^*$.

(b) Compute the feasible direction cone $C_{\mathbf{x}^*}$.

(c) Compute $\nabla f(\mathbf{x}^*)$ and verify $\nabla f(\mathbf{x}^*)^T \mathbf{d} \ge 0$ for all $\mathbf{d} \in C_{\mathbf{x}^*}$.

(d) Show that $-\nabla f(\mathbf{x}^*)$ is a nonnegative combination of active constraint normals.

(e) Write the KKT system and verify that $\mathbf{x}^*$ satisfies it.

5. **Active Set Method**

Learning objective: Understand how the KKT conditions lead to the active set algorithm.

$$
\begin{aligned}
\underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) = x_1^2 + x_2^2 - 4x_1 - 4x_2 \\
\text{subject to} \quad & x_1 + x_2 \le 3, \\
& x_1 \ge 0, \\
& x_2 \ge 0
\end{aligned}
$$

Start with working set $W = \emptyset$.

(a) Solve the subproblem corresponding to $W = \emptyset$ and determine feasibility.

(b) Add a violated constraint to W and solve the equality-constrained subproblem.

(c) Compute the multiplier(s) for the active constraint(s).

(d) Determine whether any constraint should be removed from W and update if needed.

(e) Verify that the final solution satisfies primal feasibility, dual feasibility, and complementary slackness.