Solutions to Homework 04

Math 482: Mathematical Methods of Operations Research (Spring 2026)

I. Let $x_{ij} = 1$ if item i is ranked before item j, and $x_{ij} = 0$ otherwise.

(a) The linear ordering problem is

$$\begin{aligned}
\text{maximize} \quad & z = 4x_{12} + x_{13} + 3x_{14} + 2x_{21} + 5x_{23} + 2x_{24} + 3x_{31} + x_{32} + 4x_{34} + x_{41} + 3x_{42} + 2x_{43} \\
\text{subject to} \quad & x_{ij} + x_{ji} = 1, \quad \forall i < j, \\
& x_{ij} + x_{jk} + x_{ki} \leq 2, \quad \forall \text{ distinct } i, j, k, \\
& x_{ij} \in \{0, 1\}, \quad \forall i \neq j
\end{aligned}$$

(b) For a ranking, every pair contributes the weight of the earlier vertex over the later one. For $(1, 2, 3, 4)$,

$$z = a_{12} + a_{13} + a_{14} + a_{23} + a_{24} + a_{34} = 4 + 1 + 3 + 5 + 2 + 4 = 19.$$

For $(2, 3, 1, 4)$,

$$z = a_{23} + a_{21} + a_{24} + a_{31} + a_{34} + a_{14} = 5 + 2 + 2 + 3 + 4 + 3 = 19.$$

For $(2, 1, 3, 4)$,

$$z = a_{21} + a_{23} + a_{24} + a_{13} + a_{14} + a_{34} = 2 + 5 + 2 + 1 + 3 + 4 = 17.$$

(c) Using $x_{ji} = 1 - x_{ij}$ for $i < j$, write the objective in terms of

$$x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}.$$

Then

$$\begin{aligned}
z &= 4x_{12} + x_{13} + 3x_{14} + 2(1 - x_{12}) + 5x_{23} + 2x_{24} + 3(1 - x_{13}) + (1 - x_{23}) + 4x_{34} \\
&\quad + (1 - x_{14}) + 3(1 - x_{24}) + 2(1 - x_{34}) \\
&= 12 + 2x_{12} - 2x_{13} + 2x_{14} + 4x_{23} - x_{24} + 2x_{34}.
\end{aligned}$$

Hence

$$z \leq 12 + 2 + 0 + 2 + 4 + 0 + 2 = 22.$$

So 22 is an upper bound.

(d) From part (b), we already have feasible rankings with value 19, so

$$z^* \geq 19.$$

To determine the optimum, evaluate all 24 rankings or argue from the simplified objective. The rankings with largest value are

$$(1, 2, 3, 4), \qquad (2, 3, 1, 4),$$

and each has value 19. No ranking attains 20 or more, so

$$z^* = 19.$$

Therefore the optimal ranking is *not* unique.

II. Recall

$$m_{ij} = a_{ij} - a_{ji}.$$

(a) The margins are

$$m_{12} = 2,\ m_{21} = -2,\ m_{13} = -2,\ m_{31} = 2,\ m_{14} = 2,\ m_{41} = -2,$$

$$m_{23} = 4,\ m_{32} = -4,\ m_{24} = -1,\ m_{42} = 1,\ m_{34} = 2,\ m_{43} = -2.$$

(b) The nonzero mixing terms are of the form $b_{ij,jk} = m_{ik} - m_{jk}$. For distinct i, j, k, these are:

$$b_{12,23} = m_{13} - m_{23} = -2 - 4 = -6, \qquad b_{12,24} = m_{14} - m_{24} = 2 - (-1) = 3,$$

$$b_{13,32} = m_{12} - m_{32} = 2 - (-4) = 6, \qquad b_{13,34} = m_{14} - m_{34} = 2 - 2 = 0,$$

$$b_{14,42} = m_{12} - m_{42} = 2 - 1 = 1, \qquad b_{14,43} = m_{13} - m_{43} = -2 - (-2) = 0,$$

$$b_{21,13} = m_{23} - m_{13} = 4 - (-2) = 6, \qquad b_{21,14} = m_{24} - m_{14} = -1 - 2 = -3,$$

$$b_{23,31} = m_{21} - m_{31} = -2 - 2 = -4, \qquad b_{23,34} = m_{24} - m_{34} = -1 - 2 = -3,$$

$$b_{24,41} = m_{21} - m_{41} = -2 - (-2) = 0, \qquad b_{24,43} = m_{23} - m_{43} = 4 - (-2) = 6,$$

$$b_{31,12} = m_{32} - m_{12} = -4 - 2 = -6, \qquad b_{31,14} = m_{34} - m_{14} = 2 - 2 = 0,$$

$$b_{32,21} = m_{31} - m_{21} = 2 - (-2) = 4, \qquad b_{32,24} = m_{34} - m_{24} = 2 - (-1) = 3,$$

$$b_{34,41} = m_{31} - m_{41} = 2 - (-2) = 4, \qquad b_{34,42} = m_{32} - m_{42} = -4 - 1 = -5,$$

$$b_{41,12} = m_{42} - m_{12} = 1 - 2 = -1, \qquad b_{41,13} = m_{43} - m_{13} = -2 - (-2) = 0,$$

$$b_{42,21} = m_{41} - m_{21} = -2 - (-2) = 0, \qquad b_{42,23} = m_{43} - m_{23} = -2 - 4 = -6,$$

$$b_{43,31} = m_{41} - m_{31} = -2 - 2 = -4, \qquad b_{43,32} = m_{42} - m_{32} = 1 - (-4) = 5.$$

Any omitted term is zero only because the formula gives 0 or because it is not of the form $b_{ij,jk}$.

(c) The QLOP is

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i \neq j} a_{ij} x_{ij} + \epsilon \sum_{i \neq j} \sum_{j \neq k} b_{ij,jk} x_{ij} x_{jk} \\ \text{subject to} & x_{ij} + x_{ji} = 1, \quad \forall i < j, \\ & x_{ij} + x_{jk} + x_{ki} \leq 2, \quad \forall \text{ distinct } i, j, k, \\ & x_{ij} \in \{0, 1\}, \quad \forall i \neq j \end{array}$$

with the a_{ij} from Problem I and the above $b_{ij,jk}$.

(d) From Problem I, the optimal LOP rankings are $(1, 2, 3, 4)$ and $(2, 3, 1, 4)$.
For $(1, 2, 3, 4)$, the consecutive pairs are $(1, 2), (2, 3), (3, 4)$, so the quadratic contribution is

$$b_{12,23} + b_{23,34} = (-6) + (-3) = -9.$$

Thus the full objective is

$$19 - 9\epsilon.$$

For $(2,3,1,4)$, the consecutive pairs are $(2,3),(3,1),(1,4)$, so the quadratic contribution is

$$b_{23,31}+b_{31,14}=(-4)+0=-4.$$

Thus the full objective is

$$19-4\epsilon.$$

(e) Since $-4\epsilon>-9\epsilon$ for every $\epsilon>0$, the ranking $(2,3,1,4)$ has larger quadratic objective than $(1,2,3,4)$. Since these were already the only LOP-optimal rankings, it follows that for sufficiently small $\epsilon>0$ the QLOP has the unique optimal ranking

$$(2,3,1,4).$$

(f) The parameter ϵ must be small enough so that the quadratic correction only breaks ties among rankings with best linear value. If ϵ were too large, then a ranking with smaller linear value could overcome that loss through the quadratic term, so the problem would no longer primarily reflect the original LOP objective.

III. (a) Introduce binary variables $y_{ij,kl}$ intended to represent $x_{ij}x_{kl}$. The linearized model is

$$\begin{aligned}
\text{maximize}\quad & \sum_{i\neq j} a_{ij}x_{ij}+\sum_{i\neq j}\sum_{k\neq l} b_{ij,kl}y_{ij,kl}\\
\text{subject to}\quad & x_{ij}+x_{ji}=1,\quad \forall i<j,\\
& x_{ij}+x_{jk}+x_{ki}\le 2,\quad \forall \text{ distinct } i,j,k,\\
& y_{ij,kl}\le x_{ij},\quad \forall i\neq j,\ k\neq l,\\
& y_{ij,kl}\le x_{kl},\quad \forall i\neq j,\ k\neq l,\\
& y_{ij,kl}\ge x_{ij}+x_{kl}-1,\quad \forall i\neq j,\ k\neq l,\\
& x_{ij}\in\{0,1\},\quad \forall i\neq j,\\
& y_{ij,kl}\in\{0,1\},\quad \forall i\neq j,\ k\neq l
\end{aligned}$$

(b) If $x_{ij}=0$, then $y_{ij,kl}\le x_{ij}=0$, so $y_{ij,kl}=0$. Similarly, if $x_{kl}=0$, then $y_{ij,kl}=0$. If $x_{ij}=x_{kl}=1$, then

$$y_{ij,kl}\ge x_{ij}+x_{kl}-1=1,$$

so $y_{ij,kl}=1$. Therefore, when the variables are binary,

$$y_{ij,kl}=1 \iff x_{ij}=1 \text{ and } x_{kl}=1,$$

which is exactly

$$y_{ij,kl}=x_{ij}x_{kl}.$$

(c) In the LP relaxation, x and y are allowed to be fractional. Then the three inequalities only describe the convex envelope of the product over $[0,1]^2$, not exact multiplication. For instance, if $x_{ij}=x_{kl}=1/2$, then $y_{ij,kl}$ can vary between 0 and $1/2$, while the true product is $1/4$. Thus the relaxation admits extra fractional points and is weaker.

(d) There are $n(n-1)=O(n^2)$ variables x_{ij}. There are

$$[n(n-1)]^2=O(n^4)$$

variables $y_{ij,kl}$. So the total number of variables is $O(n^4)$.

For constraints, there are

$$\binom{n}{2} = O(n^2)$$

tournament constraints and $O(n^3)$ 3-cycle constraints. The linearization contributes $3\,n^2(n-1)^2 = O(n^4)$ inequalities. Hence the total number of constraints is dominated by $O(n^4)$.

(e) The model is much harder because it introduces $O(n^4)$ new variables and constraints, and its LP relaxation is weaker. So each LP is larger, the bounds are poorer, and branch-and-bound or cutting-plane methods typically require much more work.

IV. The objective is

$$(x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + (x_1 - 2)^2 + x_2^2 + (x_3 - 1)^2 + (x_4 - 3)^2.$$

(a) Expanding,

$$\begin{aligned} f(\mathbf{x}) &= (x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_3^2) + (x_3^2 - 2x_3x_4 + x_4^2) \\ &\quad + (x_1^2 - 4x_1 + 4) + x_2^2 + (x_3^2 - 2x_3 + 1) + (x_4^2 - 6x_4 + 9) \\ &= 2x_1^2 + 3x_2^2 + 3x_3^2 + 2x_4^2 - 2x_1x_2 - 2x_2x_3 - 2x_3x_4 \\ &\quad - 4x_1 - 2x_3 - 6x_4 + 14. \end{aligned}$$

Thus

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} + d$$

with

$$Q = \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 6 & -2 & 0 \\ 0 & -2 & 6 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} -4 \\ 0 \\ -2 \\ -6 \end{bmatrix}, \qquad d = 14.$$

(b) As above,

$$Q = \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 6 & -2 & 0 \\ 0 & -2 & 6 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} -4 \\ 0 \\ -2 \\ -6 \end{bmatrix}.$$

(c) The matrix Q is symmetric. By Sylvester's criterion, compute the leading principal minors:

$$\Delta_1 = 4 > 0, \qquad \Delta_2 = \det\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} = 20 > 0,$$

$$\Delta_3 = 104 > 0, \qquad \Delta_4 = 336 > 0.$$

Hence Q is positive definite.

(d) Solve $Q\mathbf{x} = -\mathbf{c}$, i.e.

$$\begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 6 & -2 & 0 \\ 0 & -2 & 6 & -2 \\ 0 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 6 \end{bmatrix}.$$

This gives the system

$$\begin{aligned} 4x_1 - 2x_2 &= 4, \\ -2x_1 + 6x_2 - 2x_3 &= 0, \\ -2x_2 + 6x_3 - 2x_4 &= 2, \\ -2x_3 + 4x_4 &= 6. \end{aligned}$$

Solving,

$$\mathbf{x}^* = \begin{bmatrix} \frac{67}{64} \\ \frac{35}{32} \\ \frac{31}{16} \\ \frac{79}{32} \end{bmatrix}.$$

(e) The minimizer is a compromise between two goals: staying near the data $(2, 0, 1, 3)$ and keeping adjacent vertices close together. So the solution smooths the data along the path graph. In particular, the extreme values are pulled toward their neighbors and the interior values are averaged with nearby vertices.

V. The objective is

$$z = x_1^2 + 2x_2^2 + x_1x_2 - 6x_1 - 4x_2.$$

(a) The gradient is

$$\nabla z = \begin{bmatrix} 2x_1 + x_2 - 6 \\ x_1 + 4x_2 - 4 \end{bmatrix}.$$

Set this equal to zero:

$$2x_1 + x_2 = 6, \qquad x_1 + 4x_2 = 4.$$

Solving gives

$$(x_1, x_2) = \left(\frac{20}{7}, \frac{2}{7}\right).$$

(b) This point is not feasible because

$$\frac{20}{7} + \frac{2}{7} = \frac{22}{7} > 3.$$

(c) Since the critical point is not feasible, the optimum occurs on the boundary of the feasible region. Because the Hessian will be positive definite, the level sets are ellipses centered at the critical point. The minimum over the feasible triangle is therefore where the smallest level set first meets the feasible region.

(d) Check the boundary pieces.
Along $x_2 = 0$ with $0 \le x_1 \le 3$,

$$z = x_1^2 - 6x_1,$$

which is minimized at $x_1 = 3$, giving $z = -9$.
Along $x_1 = 0$ with $0 \le x_2 \le 3$,

$$z = 2x_2^2 - 4x_2,$$

which is minimized at $x_2 = 1$, giving $z = -2$.

Along $x_1 + x_2 = 3$, write $x_2 = 3 - x_1$ with $0 \le x_1 \le 3$:

$$\begin{aligned} z(x_1, 3 - x_1) &= x_1^2 + 2(3 - x_1)^2 + x_1(3 - x_1) - 6x_1 - 4(3 - x_1) \\ &= 2x_1^2 - 11x_1 + 6. \end{aligned}$$

This is minimized at

$$x_1 = \frac{11}{4}, \qquad x_2 = \frac{1}{4},$$

with value

$$z^* = 2\left(\frac{11}{4}\right)^2 - 11\left(\frac{11}{4}\right) + 6 = -\frac{73}{8}.$$

Therefore the optimal solution is

$$\left(\frac{11}{4}, \frac{1}{4}\right)$$

with objective value

$$z^* = -\frac{73}{8}.$$

(e) The Hessian matrix is

$$H = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$

Its characteristic polynomial is

$$\lambda^2 - 6\lambda + 7,$$

so the eigenvalues are

$$\lambda = 3 \pm \sqrt{2},$$

both positive. Hence H is positive definite, so the objective is strictly convex. Therefore any feasible point satisfying the boundary minimization condition is the unique global minimizer.