Homework 01 — Solutions

Math 482: Mathematical Methods of Operations Research (Spring 2026)
Week 1 (Jan 12–Jan 16, 2026)

Relevant topics: Linear Programming, Standard Form, Dual, Weak Duality Theorem

I. Consider the following primal LP

$$\begin{aligned} \text{maximize} \quad & z = 3x_1 + 4x_2 \\ \text{subject to} \quad & 2x_1 - 3x_2 \le 3, \\ & 4x_1 + x_2 \le 6, \\ & x_1 + x_2 \le 5, \\ & x_i \ge 0, \ \forall i \in \{1, 2\} \end{aligned}$$

a. **Matrices A, b, and c.** Writing the constraints as $A\mathbf{x} \le \mathbf{b}$ with $\mathbf{x} = (x_1, x_2)^T$,

$$A = \begin{pmatrix} 2 & -3 \\ 4 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 6 \\ 5 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

b. **Dual problem.** The primal is a maximization with "$\le$" constraints and $\mathbf{x} \ge 0$, so the dual is a minimization with $\mathbf{y} \ge 0$ and $A^T\mathbf{y} \ge \mathbf{c}$:

$$\begin{aligned} \text{minimize} \quad & w = 3y_1 + 6y_2 + 5y_3 \\ \text{subject to} \quad & 2y_1 + 4y_2 + y_3 \ge 3, \\ & -3y_1 + y_2 + y_3 \ge 4, \\ & y_i \ge 0, \ \forall i \in \{1, 2, 3\} \end{aligned}$$

c. **Prove $z \le w$ for this particular primal–dual pair (weak duality written out).**
Let $\mathbf{x} = (x_1, x_2)^T$ be any feasible solution of the primal and let $\mathbf{y} = (y_1, y_2, y_3)^T$ be any feasible solution of the dual.
Because $\mathbf{y} \ge 0$ and $A\mathbf{x} \le \mathbf{b}$, we may multiply each primal inequality by the corresponding y_i and add:

$$y_1(2x_1 - 3x_2) + y_2(4x_1 + x_2) + y_3(x_1 + x_2) \le 3y_1 + 6y_2 + 5y_3.$$

Regroup the left-hand side by x_1 and x_2:

$$(2y_1 + 4y_2 + y_3)x_1 + (-3y_1 + y_2 + y_3)x_2 \le 3y_1 + 6y_2 + 5y_3.$$

Because $\mathbf{x} \geq 0$ and the dual constraints say

$$2y_1 + 4y_2 + y_3 \geq 3, \qquad -3y_1 + y_2 + y_3 \geq 4,$$

we can replace the coefficients on the left by the smaller numbers 3 and 4:

$$3x_1 + 4x_2 \leq (2y_1 + 4y_2 + y_3)x_1 + (-3y_1 + y_2 + y_3)x_2 \leq 3y_1 + 6y_2 + 5y_3.$$

Thus $z = 3x_1 + 4x_2 \leq 3y_1 + 6y_2 + 5y_3 = w$.

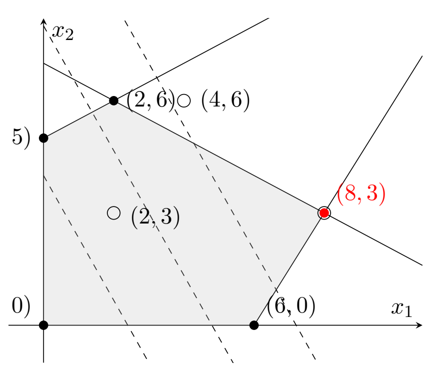
II. Consider the following primal LP

$$\begin{aligned} \text{maximize} \quad & z = 5x_1 + 3x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \leq 14, \\ & 3x_1 - 2x_2 \leq 18, \\ & -x_1 + 2x_2 \leq 10, \\ & x_i \geq 0, \ \forall i \in \{1,2\} \end{aligned}$$

a. **Graph of constraints (and feasible region).** Plot the boundary lines

$$x_1 + 2x_2 = 14, \qquad 3x_1 - 2x_2 = 18, \qquad -x_1 + 2x_2 = 10,$$

plus the axes. The feasible region is the intersection of the corresponding half-spaces with the first quadrant.



b. **Add the lines** $z = 12, 24, 36$**.** These are

$$5x_1 + 3x_2 = 12, \qquad 5x_1 + 3x_2 = 24, \qquad 5x_1 + 3x_2 = 36.$$

They are parallel, with slope $-5/3$.

c. **Add the given points.** Plot $(2,3)$, $(4,6)$, and $(8,3)$.

d. **Find $\mathbf{x}^*$ and z^*.** The vertices of the feasible region are

$$(0,0),\ (0,5),\ (2,6),\ (6,0),\ (8,3).$$

Evaluating $z = 5x_1 + 3x_2$ gives

(x_1, x_2)	$(0,0)$	$(0,5)$	$(2,6)$	$(6,0)$	$(8,3)$
z	0	15	28	30	49

so

$$\mathbf{x}^* = (8,3)^T, \qquad z^* = 49.$$

e. **Write the dual.** Since the primal is a maximization problem with "$\leq$" constraints and $\mathbf{x} \geq 0$, the dual is

$$\begin{aligned} \text{minimize} \quad & w = 14y_1 + 18y_2 + 10y_3 \\ \text{subject to} \quad & y_1 + 3y_2 - y_3 \geq 5, \\ & 2y_1 - 2y_2 + 2y_3 \geq 3, \\ & y_i \geq 0,\ \forall i \in \{1,2,3\} \end{aligned}$$

f. **Find $\mathbf{y}^*$ and w^*.** At $\mathbf{x}^* = (8,3)$, the first two constraints are tight and the third is slack:

$$8 + 2 \cdot 3 = 14, \qquad 3 \cdot 8 - 2 \cdot 3 = 18, \qquad -8 + 2 \cdot 3 = -2 < 10.$$

By complementary slackness, $y_3^* = 0$. Since $x_1^* > 0$ and $x_2^* > 0$, the dual inequalities are tight at optimum:

$$y_1 + 3y_2 - y_3 = 5, \qquad 2y_1 - 2y_2 + 2y_3 = 3.$$

With $y_3 = 0$, this becomes $y_1 + 3y_2 = 5$ and $y_1 - y_2 = \frac{3}{2}$, hence

$$y_2^* = \frac{7}{8}, \qquad y_1^* = \frac{19}{8}, \qquad y_3^* = 0.$$

Then

$$w^* = 14 \cdot \frac{19}{8} + 18 \cdot \frac{7}{8} = 49 = z^*.$$

III. Consider the following primal LP.

$$\begin{aligned} \text{maximize} \quad & z = -2x_2 \\ \text{subject to} \quad & x_1 + 4x_2 - x_3 \leq 1, \\ & -2x_1 - 3x_2 + x_3 \leq -2, \\ & 4x_1 + x_2 - x_3 \leq 1, \\ & x_i \geq 0,\ \forall i \in \{1,2,3\} \end{aligned}$$

a. **Write the dual.** The primal has "$\leq$" constraints and $\mathbf{x} \geq 0$, so the dual is

$$\begin{aligned} \text{minimize} \quad & w = y_1 - 2y_2 + y_3 \\ \text{subject to} \quad & y_1 - 2y_2 + 4y_3 \geq 0, \\ & 4y_1 - 3y_2 + y_3 \geq -2, \\ & -y_1 + y_2 - y_3 \geq 0, \\ & y_i \geq 0, \ \forall i \in \{1, 2, 3\} \end{aligned}$$

b. **Show $\mathbf{y} = (2t, 3t, t)^T$ is dual feasible for all $t \geq 0$.** Let $t \geq 0$ and set $\mathbf{y} = (2t, 3t, t)^T$. Then $\mathbf{y} \geq 0$, and

$$y_1 - 2y_2 + 4y_3 = 2t - 6t + 4t = 0 \geq 0,$$

$$4y_1 - 3y_2 + y_3 = 8t - 9t + t = 0 \geq -2,$$

$$-y_1 + y_2 - y_3 = -2t + 3t - t = 0 \geq 0,$$

so $\mathbf{y}$ is feasible.

c. **Dual unbounded and primal infeasible.** For $\mathbf{y} = (2t, 3t, t)^T$, the dual objective is

$$w(t) = y_1 - 2y_2 + y_3 = 2t - 2(3t) + t = -3t \to -\infty \quad \text{as } t \to \infty,$$

so the dual is unbounded below.

If the primal had a feasible solution $\mathbf{x}$, then by weak duality we would have $z(\mathbf{x}) \leq w(\mathbf{y})$ for every dual feasible $\mathbf{y}$. Taking $\mathbf{y} = (2t, 3t, t)^T$ and letting $t \to \infty$ would force $z(\mathbf{x}) \leq -3t$ for all $t \geq 0$, which is impossible. Hence the primal is infeasible.