Equality-Constrained Quadratic Programming

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1 Introduction

In previous lectures, we studied quadratic objectives of the form

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x},$$

and saw that optimal solutions in the unconstrained setting occur at critical points satisfying

$$Q\mathbf{x} = -\mathbf{c}.$$

We now consider what happens when $\mathbf{x}$ is required to satisfy linear equality constraints.

In this lecture, we will focus on equality-constrained quadratic programs in the following form

$$\begin{aligned} &\text{minimize} \quad && \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} \\ &\text{subject to} \quad && A\mathbf{x} = \mathbf{b} \end{aligned}$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric, $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$.

2 Geometry of the feasible set

The constraint $A\mathbf{x} = \mathbf{b}$ is either infeasible, represents a unique point, or defines an affine subspace of $\mathbb{R}^n$. In the latter case, the affine subspace of solutions is given by

$$\mathbf{x}_p + \text{nul}(A),$$

where $\mathbf{x}_p$ is a particular solution to $A\mathbf{x} = \mathbf{b}$ and $\text{nul}(A)$ is the nullspace of A. Thus, the feasible set is a translation of the null space and we are minimizing $f(\mathbf{x}_p + \mathbf{d})$ over $\mathbf{d} \in \text{nul}(A)$.

2.1 Feasible directions and optimality

A vector $\mathbf{d}$ is a feasible direction if $A\mathbf{d} = \mathbf{0}$. These are precisely the directions along which we may move without violating the constraints.

At an optimal point $\mathbf{x}^*$, the objective cannot decrease in any feasible direction. Thus,

$$\left.\frac{d}{dt}f(\mathbf{x}^* + t\mathbf{d})\right|_{t=0} = 0,$$

for all $\mathbf{d} \in \text{nul}(A)$. Using the chain rule, we have

$$\frac{d}{dt}f(\mathbf{x}^* + t\mathbf{d}) = \nabla f(\mathbf{x}^* + t\mathbf{d})^T\mathbf{d}.$$

Since $\nabla f(\mathbf{x}) = Q\mathbf{x} + \mathbf{c}$, it follows that

$$\frac{d}{dt} f(\mathbf{x}^* + t\mathbf{d})\Big|_{t=0} = (Q\mathbf{x}^* + \mathbf{c})^T \mathbf{d} = 0,$$

for all $\mathbf{d} \in \text{nul}(A)$. Therefore,

$$Q\mathbf{x}^* + \mathbf{c} \perp \text{nul}(A).$$

Recall that

$$\text{nul}(A)^\perp = \text{row}(A) = \text{col}(A^T) = \text{col}(-A^T).$$

Therefore, there exists $\mathbf{y} \in \mathbb{R}^m$ such that

$$Q\mathbf{x}^* + \mathbf{c} = -A^T \mathbf{y}.$$

This is the first-order optimality condition. We write the condition like this so the resulting KKT system aligns with the system we derive for inequality-constrained QPs.

3 The KKT system

The first-order optimality condition is a necessary condition on when a feasible vector $\mathbf{x}$ is optimal. Combining the optimality condition with feasibility yields

$$\begin{aligned} Q\mathbf{x} + \mathbf{c} + A^T\mathbf{y} &= 0, \\ A\mathbf{x} - \mathbf{b} &= 0, \end{aligned}$$

which is the KKT system. This system can be written as a matrix equation

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} -\mathbf{c} \\ \mathbf{b} \end{bmatrix}.$$

Therefore, solving the equality-constrained QP reduces to solving the KKT system when an optimal solution exists.

Example

Consider the following QP

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\mathbf{x}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 2 \\ 0 \end{bmatrix}^T \mathbf{x} \\ \text{subject to} & \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x} = 1 \end{array}$$

The KKT system is

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

Solving yields

$$\mathbf{x}^* = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} \qquad y_1 = -1/2.$$

Note that $\mathbf{x}^*$ is an optimal solution corresponding to objective value of $z = -5/4$. Moreover, $\mathbf{x}^*$ is not the unconstrained minimizer, which is $x_1 = 4/3$ and $x_2 = -2/3$ but is not feasible. The multiplier $y_1 = -1/2$ tells us that the KKT solution does not match the global minimizer.

Example

Consider the following QP

$$\begin{aligned} \text{minimize} \quad & x_1^2 - x_2^2 \\ \text{subject to} \quad & x_1 + x_2 = 0 \end{aligned}$$

The KKT system is

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving yields infinitely many solutions. In fact, the feasibility condition gives $x_2 = -x_1$, and then the optimality conditions reduce to

$$y_1 = -2x_1 = 2x_2.$$

Thus, every feasible point satisfies the KKT system. Moreover, along the feasible set we have

$$x_1^2 - x_2^2 = x_1^2 - (-x_1)^2 = 0.$$

Therefore, every feasible point is optimal, and the optimal objective value is $z = 0$. If we select $x_1 = x_2 = 0$, then $y_1 = 0$ which means the KKT solution is the critical point of the quadratic (not a global minimizer since the quadratic is indefinite). If we select $x_1 = 1$, $x_2 = -1$, then $y_1 = -2$ which means the KKT solution is not the critical point of the quadratic.

Example

Consider the following QP

$$\begin{aligned} \text{minimize} \quad & x_1^2 - x_2^2 \\ \text{subject to} \quad & x_1 = 0 \end{aligned}$$

The KKT System is

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving yields

$$x_1 = 0, \qquad x_2 = 0, \qquad y_1 = 0.$$

The $y_1 = 0$ suggests that the KKT solution is the critical point of the quadratic. However, this solution is not optimal. In fact, the condition $x_1 = 0$ implies that the quadratic is $-x_2^2$ which tends to $-\infty$.

3.1 Existence and uniqueness

Recall that in the unconstrained case, convexity depends on the eigenvalues of Q. In the constrained case, we restrict attention to directions $\mathbf{d}$ satisfying $A\mathbf{d} = 0$.

We say Q is positive definite on $\text{nul}(A)$ if

$$\mathbf{d}^T Q \mathbf{d} > 0,$$

for all non-zero $\mathbf{d} \in \text{nul}(A)$.

Theorem 3.1. *Suppose that Q is positive definite on* $\text{nul}(A)$. *If $A\mathbf{x} = \mathbf{b}$ is feasible, then the QP has a unique minimizer.*

Proof. Let $\mathbf{x}_p \in \mathbb{R}^n$ denote a feasible solution, that is, $A\mathbf{x}_p = \mathbf{b}$. Then, every feasible point can be written in the form

$$\mathbf{x} = \mathbf{x}_p + \mathbf{d},$$

where $\mathbf{d} \in \text{nul}(A)$. Thus, minimizing the objective over all feasible solutions is equivalent to minimizing

$$\phi(\mathbf{d}) = f(\mathbf{x}_p + \mathbf{d})$$

over $\mathbf{d} \in \text{nul}(A)$. By expanding $\phi(\mathbf{d})$ and using the symmetry of Q, we have

$$\begin{aligned}\phi(\mathbf{d}) &= \frac{1}{2}(\mathbf{x}_p + \mathbf{d})^T Q(\mathbf{x}_p + \mathbf{d}) + \mathbf{c}^T(\mathbf{x}_p + \mathbf{d}) \\ &= \frac{1}{2}\mathbf{d}^T Q\mathbf{d} + (Q\mathbf{x}_p + \mathbf{c})^T\mathbf{d} + \left(\frac{1}{2}\mathbf{x}_p^T Q\mathbf{x}_p + \mathbf{c}^T\mathbf{x}_p\right)\end{aligned}$$

Therefore, ϕ is a quadratic function on the subspace $\text{nul}(A)$ whose quadratic part is

$$\frac{1}{2}\mathbf{d}^T Q\mathbf{d}.$$

Because Q is positive definite on $\text{nul}(A)$, we have

$$\mathbf{d}^T Q\mathbf{d} > 0,$$

for all $\mathbf{d} \in \text{nul}(A) \setminus \{\mathbf{0}\}$. So, ϕ is strictly convex on $\text{nul}(A)$ and therefore has a unique minimum. □

Corollary 3.2. *Suppose that Q is positive definite on $\mathbb{R}^n$. If $A\mathbf{x} = \mathbf{b}$ is feasible, then the QP has a unique minimizer.*

The following example illustrates a case where Q is positive semi-definite on $\mathbb{R}^n$ but positive definite on $\text{nul}(A)$ so there is still a unique minimizer of the QP.

Example

Consider the QP with

$$Q = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenvalues of Q are $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 3$. Therefore, Q is positive semidefinite on $\mathbb{R}^3$. The rank of A is 2, so the rank-nullity theorem implies that the null space of A has dimension 1. In fact,

$$\text{nul}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}.$$

Therefore, every non-zero null vector of A is of the form

$$\mathbf{d} = t\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \qquad t \neq 0.$$

Note that $\mathbf{d}^T Q\mathbf{d} = 2t^2 > 0$ for all $t \neq 0$. Hence, Q is positive definite on $\text{nul}(A)$, so the QP has a unique minimizer. We identify the minimizer using the KKT system. The feasibility conditions state that $A\mathbf{x} = \mathbf{b}$, which in this case give us

$$x_2 = 1 \qquad x_1 + x_3 = 0.$$

The optimality conditions state that $Q\mathbf{x} + \mathbf{c} = A^T\mathbf{y}$, which in this case give us

$$\begin{aligned} x_1 - x_2 - 2 + y_2 &= 0, \\ -x_1 + 2x_2 - x_3 + y_1 &= 0, \\ -x_2 + x_3 + y_2 &= 0. \end{aligned}$$

Equating the first and third equation gives

$$x_1 - x_2 - 2 = -x_2 + x_3.$$

Since $x_2 = 1$ and $x_3 = -x_1$, we have that $x_1 = 1$ and $x_3 = -1$. Therefore, the solution to the KKT system is

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}.$$