Duality of Quadratic Programs

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1 Introduction

We are focused on equality-constrained quadratic programs of the form

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{b} \end{aligned}$$

where $Q \in \mathbb{R}^{n\times n}$ is symmetric, $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$.

The KKT system encodes the necessary conditions for optimality and feasibility:

$$\begin{aligned} Q\mathbf{x} + \mathbf{c} + A^T\mathbf{y} &= 0, \\ A\mathbf{x} - \mathbf{b} &= 0. \end{aligned}$$

If Q is positive definite on $\text{nul}(A)$, then these conditions are necessary and sufficient.

The Lagrangian function encodes the KKT system:

$$L(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\mathbf{x}^T Q\mathbf{x} + \mathbf{c}^T\mathbf{x} + \mathbf{y}^T(A\mathbf{x} - \mathbf{b}).$$

In particular, the gradient of the Lagragian is given by

$$\begin{aligned} \nabla_{\mathbf{x}} L(\mathbf{x}, \mathbf{y}) &= Q\mathbf{x} + \mathbf{c} + A^T\mathbf{y}, \\ \nabla_{\mathbf{y}} L(\mathbf{x}, \mathbf{y}) &= A\mathbf{x} - \mathbf{b}. \end{aligned}$$

Hence the critical points of the Lagrangian encode the KKT system. In this lecture, we use the Lagrangian to derive the *dual problem*, to prove weak and strong duality in the quadratic setting, and to explain how the multipliers encode information about the constrained problem.

2 The Dual

For a fixed $\mathbf{y}$, the function $L(\mathbf{x}, \mathbf{y})$ is an unconstrained quadratic in $\mathbf{x}$. We define the *dual function* by

$$g(\mathbf{y}) = \inf_{\mathbf{x}\in\mathbb{R}^n} L(\mathbf{x}, \mathbf{y}),$$

where the infimum inf denotes the greatest lower bound.

Example

Consider the function

$$f(x) = 1 - x^2, \qquad -1 < x < 1.$$

On the open interval $(-1, 1)$, $f(x)$ does not obtain a minimum as the values of $f(x)$ get arbitrarily close to 0 but never reach 0. In contrast, the infimum is attained and $\inf_{-1<x<1} f(x) = 0$ since no value greater than 0 can serve as a lower bound of $f(x)$. □

The situation in the previous example will not occur with equality-constrained QPs over $\mathbb{R}^n$. However, the infimum still plays an important role as illustrated by the following example.

Example

Consider the Lagrangian function

$$L(\mathbf{x}, \mathbf{y}) = x_1^2 - x_2^2 + y_1 (x_1 + x_2).$$

Then, the dual function is

$$\begin{aligned} g(y_1) &= \inf_{x_1, x_2} \left(x_1^2 - x_2^2 + y_1 x_1 + y_1 x_2\right) \\ &= \inf_{x_1} \left(x_1^2 + y_1 x_1\right) + \inf_{x_2} \left(-x_2^2 + y_1 x_2\right) \\ &= -\frac{1}{4} y_1^2 - \infty = -\infty, \end{aligned}$$

for all $y_1 \in \mathbb{R}$. □

With equality-constrained QPs over $\mathbb{R}^n$, the only obstruction to evaluating the dual function is not failure of attainment, but rather unboundedness. This explains the role of the infimum very clearly: for each $\mathbf{y}$, either the Lagrangian is bounded below and the infimum is the minimum, or the Lagrangian is unbounded below and the dual function takes the value $-\infty$.

The dual problem is defined as follows

$$\sup_{\mathbf{y} \in \mathbb{R}^m} g(\mathbf{y}),$$

where the supremum sup is the least upper bound. Note that, analogous to the infimum, we are using the supremum to handle unboundedness. Moreover, the dual is defined only in the multiplier variables. The primal problem minimizes over $\mathbf{x}$ subject to the constraints. The dual problem leaves the constraints inside the Lagrangian, minimizes over $\mathbf{x}$ first, and then asks for the maximum produced by the multipliers.

Example

Consider the following QP

$$\begin{aligned} &\text{minimize} && \frac{1}{2}\mathbf{x}^T \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x} - \begin{bmatrix} 2 \\ 0 \end{bmatrix}^T \mathbf{x} \\ &\text{subject to} && \begin{bmatrix} 1 & 1 \end{bmatrix} \mathbf{x} = 1 \end{aligned}$$

The Lagrangian is given by

$$L(\mathbf{x}, \mathbf{y}) = x_1^2 + x_2^2 + x_1 x_2 - 2x_1 + y_1 (x_1 + x_2) - y_1.$$

To find the dual function, we complete the square on the Lagrangian:

$$
\begin{aligned}
L(\mathbf{x},\mathbf{y}) &= x_1^2 + x_2^2 + x_1x_2 - 2x_1 + y_1\,(x_1+x_2) - y_1 \\
&= \left(x_1^2 + (x_2 - 2 + y_1)x_1\right) + \left(x_2^2 + y_1x_2\right) - y_1 \\
&= \left(x_1 + \frac{x_2 - 2 + y_1}{2}\right)^2 - \frac{(x_2-2+y_1)^2}{4} + \left(x_2^2 + y_1x_2\right) - y_1 \\
&= \left(x_1 + \frac{x_2 - 2 + y_1}{2}\right)^2 + \left(\frac{3}{4}x_2^2 + \frac{1}{2}(y_1+2)x_2\right) - \frac{1}{4}y_1^2 - 1 \\
&= \left(x_1 + \frac{x_2 - 2 + y_1}{2}\right)^2 + \frac{3}{4}\left(x_2 + \frac{y_1+2}{3}\right)^2 - \frac{3}{4}\frac{(y_1+2)^2}{9} - \frac{1}{4}y_1^2 - 1 \\
&= \left(x_1 + \frac{x_2 - 2 + y_1}{2}\right)^2 + \frac{3}{4}\left(x_2 + \frac{y_1+2}{3}\right)^2 - \frac{1}{3}y_1^2 - \frac{1}{3}y_1 - \frac{4}{3}.
\end{aligned}
$$

The infimum in x_1, x_2 occurs when the two quadratic terms are zero, that is, when $x_1 = -(x_2 - 2 + y_1)/2$ and $x_2 = -(y_1+2)/3$. Substitution gives

$$
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 4 - y_1 \\ -y_1 - 2 \end{bmatrix}.
$$

At the infimum, the dual function is given by

$$
g(y_1) = -\frac{1}{3}y_1^2 - \frac{1}{3}y_1 - \frac{4}{3}.
$$

This quadratic is concave and therefore attains a global maximum, which we determine using the derivative:

$$
g'(y_1) = -\frac{2}{3}y_1 - \frac{1}{3}.
$$

Hence, the maximum occurs at the value $y_1 = -1/2$. At this value, $g(-1/2) = -5/4$. Moreover, this multiplier provides the optimal solution in the problem variables:

$$
\mathbf{x}^* = \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix},
$$

with a corresponding objective value of $f(\mathbf{x}^*) = -5/4$. □

3 Weak and Strong Duality

In this section, we introduce the notion of weak and strong duality for QPs. The following theorem handles weak duality for QPs.

Theorem 3.1 (Weak Duality)**.** *Suppose that the QP is feasible. If $\overline{\mathbf{x}}$ is a feasible solution, then*

$$
g(\mathbf{y}) \le f(\overline{\mathbf{x}}).
$$

for all $\mathbf{y} \in \mathbb{R}^m$.

Proof. Suppose that $\overline{\mathbf{x}}$ is feasible. Then,

$$
L(\overline{\mathbf{x}},\mathbf{y}) = f(\overline{\mathbf{x}}),
$$

for all $\mathbf{y} \in \mathbb{R}^m$. By definition of the infimum,

$$g(\mathbf{y}) = \inf_{\mathbf{x} \in \mathbb{R}^n} L(\mathbf{x}, \mathbf{y}) \leq L(\overline{\mathbf{x}}, \mathbf{y}) = f(\overline{\mathbf{x}}).$$

$\square$

Thus every choice of multipliers gives a lower bound on the value of every primal feasible point. This is the first major reason the dual is important. It does not merely provide another optimization problem, it provides certificates in the form of lower bounds.

The following theorem handles strong duality for convex equality-constrained QPs.

Theorem 3.2 (Strong Duality)**.** *Suppose that the QP is feasible and Q is positive semidefinite. Then, for any solution $(\mathbf{x}^*, \mathbf{y}^*)$ to the KKT system, we have*

$$g(\mathbf{y}^*) = f(\mathbf{x}^*).$$

Moreover, $\mathbf{x}^*$ is an optimal solution to the QP.

Proof. Suppose that $(\mathbf{x}^*, \mathbf{y}^*)$ is a solution to the KKT system. Since $A\mathbf{x}^* = \mathbf{b}$, we have

$$L(\mathbf{x}^*, \mathbf{y}) = f(\mathbf{x}^*),$$

for all $\mathbf{y} \in \mathbb{R}^m$. In particular, weak duality states that $g(\mathbf{y}) \leq f(\mathbf{x}^*)$, for all $\mathbf{y} \in \mathbb{R}^m$.

Now, the gradient of $L(\mathbf{x}, \mathbf{y}^*)$, with respect to $\mathbf{x}$, is given by

$$\nabla_{\mathbf{x}} L(\mathbf{x}, \mathbf{y}^*) = Q\mathbf{x} + \mathbf{c} + A^T \mathbf{y}^*.$$

Hence, $\mathbf{x}^*$ is a critical point of $L(\mathbf{x}, \mathbf{y}^*)$. Moreover, the Hessian of $L(\mathbf{x}, \mathbf{y}^*)$, with respect to $\mathbf{x}$, is given by

$$\nabla^2_{\mathbf{xx}} L(\mathbf{x}, \mathbf{y}^*) = Q,$$

which is positive semidefinite. Therefore, $L(\mathbf{x}, \mathbf{y}^*)$ is convex and the infimum (minimum) is attained at $\mathbf{x}^*$.

Hence,

$$g(\mathbf{y}^*) = \inf_{\mathbf{x} \in \mathbb{R}^n} L(\mathbf{x}, \mathbf{y}^*) = L(\mathbf{x}^*, \mathbf{y}^*) = f(\mathbf{x}^*).$$

If $\mathbf{x}^*$ were not optimal, then there exists a feasible $\overline{\mathbf{x}}$ such that $f(\overline{\mathbf{x}}) < f(\mathbf{x}^*)$. However, this contradicts weak duality since $g(\mathbf{y}^*) = f(\mathbf{x}^*) > f(\overline{\mathbf{x}})$. $\square$

4 The Existence of Optimal Solutions

It is worth noting that strong duality does not guarantee the KKT system will have a solution when Q is positive semidefinite, only that when a solution exists it corresponds to an optimal solution and the dual gap is zero. The following example illustrates this point.

Example

Consider the QP with

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \end{bmatrix}.$$

Note that the matrix Q is positive semidefinite. However, the KKT system (in matrix form) is given by

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

which is infeasible. From the Lagrangian perspective, we have

$$\begin{aligned} g(y_1) &= \inf_{x_1, x_2} \left(x_1^2 - x_2 + y_1 x_1 \right) \\ &= \inf_{x_1} \left(x_1^2 + y_1 x_1 \right) + \inf_{x_2} (-x_2) \\ &= -\frac{1}{4} y_1^2 - \infty = -\infty, \end{aligned}$$

for all $y_1 \in \mathbb{R}$. Therefore, the dual function is not finite for any multiplier. □

It is worth contrasting this point with the sufficient conditions of optimality, which state that when Q is positive definite along the feasible directions there is a unique minimizer. In the positive semidefinite case, we have the following result.

Theorem 4.1. *Suppose that the QP is feasible and Q is positive semidefinite. Let $L(\mathbf{x}, \mathbf{y})$ denote the Lagrangian function and $g(\mathbf{y}) = \inf_{\mathbf{x} \in \mathbb{R}^n} L(\mathbf{x}, \mathbf{y})$ denote the dual function. Then, $g(\mathbf{y})$ is finite if and only if*

$$\mathbf{c} + A^T \mathbf{y} \in \operatorname{col}(Q).$$

Proof. Define $\mathbf{d} = \mathbf{c} + A^T \mathbf{y}$. Then, the Lagrangian function can be written as

$$L(\mathbf{x}, \mathbf{y}) = \frac{1}{2} \mathbf{x}^T Q \mathbf{x} + \mathbf{d}^T \mathbf{x} + \mathbf{y}^T \mathbf{b}.$$

Since Q is positive semidefinite, the Lagrangian is bounded below if and only if the linear term $\mathbf{d}^T \mathbf{x}$ vanishes for each $\mathbf{x} \in \operatorname{nul}(Q)$. That is, the Lagrangian is bounded below if and only if $\mathbf{d} \in \operatorname{nul}(Q)^{\perp}$. Since Q is symmetric, it follows that $\operatorname{nul}(Q)^{\perp} = \operatorname{col}(Q)$. Therefore, the Lagrangian is bounded below if and only if

$$\mathbf{c} + A^T \mathbf{y} \in \operatorname{col}(Q).$$

□

Example

Consider the QP with

$$Q = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}.$$

Since Q is positive semidefinite, the dual function is finite if and only if $\mathbf{c}+A^T\mathbf{y} \in \text{col}Q = \text{nul}(Q)^{\perp}$. In this case, we require $y_1+2y_2 = 2$. The first order optimality condition requires $Q\mathbf{x} = -A^T\mathbf{y}-\mathbf{c}$. This system has a solution when $y_1+2y_2 = 2$, in which case

$$-A^T\mathbf{y}-\mathbf{c} = \begin{bmatrix} 2-y_2 \\ -y_1 \\ -y_2 \end{bmatrix} = \begin{bmatrix} 2-y_2 \\ 2y_2-2 \\ -y_2 \end{bmatrix}.$$

The eigenvalues of Q are $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 3$, with corresponding normalized eigenvectors

$$\mathbf{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Let D denote diagonal matrix with eigenvalues of Q and $V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$. We solve the system $D\mathbf{z} = V^T\left(-A^T\mathbf{y}-\mathbf{c}\right)$ to obtain (note that z_1 is a free variable)

$$z_1 = \sqrt{3}t, \qquad z_2 = -\frac{2}{\sqrt{2}}, \qquad z_3 = \frac{6-6y_2}{3\sqrt{6}}.$$

Then, we compute $\mathbf{x} = V\mathbf{z}$

$$\begin{aligned} \mathbf{x} &= V\mathbf{z} \\ &= \sqrt{3}t\mathbf{v}_1 - \frac{2}{\sqrt{2}}\mathbf{v}_2 + \frac{6-6y_2}{3\sqrt{6}}\mathbf{v}_3 \\ &= t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{1-y_2}{3}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 4-y_2 \\ -2+2y_2 \\ -2-y_2 \end{bmatrix}. \end{aligned}$$

If we substitute this $\mathbf{x}$ into the Lagrangian, we obtain the dual function

$$g(\mathbf{y}) = L(\mathbf{x},\mathbf{y}) = \begin{cases} -y_2^2+4y_2-4 & \text{if } y_1+2y_2 = 2, \\ -\infty & \text{otherwise.} \end{cases}$$

Hence, the dual problem is to maximize $-y_2^2+4y_2-4$, which occurs when $y_2 = 2$. At this multiplier, we have $y_1 = -2$, and

$$\mathbf{x} = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}.$$

This vector is feasible when $t = 1/3$. In that case, we get an optimal solution of

$$\mathbf{x}^* = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

with corresponding objective value $f(\mathbf{x}^*) = g(2) = 0$. □

We conclude with the following result which connects finite dual functions to the existence of optimal solutions.

Theorem 4.2. *Suppose that the QP is feasible and Q is positive semidefinite. Then, the QP has an optimal solution if and only if there exists $\mathbf{y} \in \mathbb{R}^m$ such that the dual function $g(\mathbf{y})$ is finite.*

Proof. Suppose that the QP has an optimal solution $\mathbf{x}^*$. Then, the first-order optimality conditions imply that there exists a $\mathbf{y}^* \in \mathbb{R}^m$ such that

$$Q\mathbf{x}^* + \mathbf{c} + A^T\mathbf{y}^* = 0, \qquad A\mathbf{x}^* - \mathbf{b} = 0.$$

Rearranging the first equation gives

$$\mathbf{c} + A^T\mathbf{y}^* = -Q\mathbf{x}^* \in \mathrm{col}(Q).$$

By Theorem 4.1, this implies that $g(\mathbf{y}^*)$ is finite.

Conversely, suppose there exists $\mathbf{y} \in \mathbb{R}^m$ such that $g(\mathbf{y})$ is finite. By Theorem 4.1, the set of all $\mathbf{y} \in \mathbb{R}^m$ such that $g(\mathbf{y})$ is finite can be defined as follows

$$\mathcal{Y} = \left\{\mathbf{y} \in \mathbb{R}^m \colon \exists \mathbf{x} \in \mathbb{R}^n \ s.t. \ \mathbf{c} + A^T\mathbf{y} = Q\mathbf{x}\right\}.$$

Since the QP is feasible, weak duality implies that the set $g(\mathcal{Y})$ is bounded above. Therefore, the completeness axiom states that $g(\mathcal{Y})$ has a least upper bound (supremum); in fact, in this case it has a maximum. Let $\mathbf{y}^*$ denote a maximum. Note that the gradient of $L(\mathbf{x}, \mathbf{y}^*)$, with respect to $\mathbf{x}$, is given by

$$\nabla_{\mathbf{x}} L(\mathbf{x}, \mathbf{y}^*) = Q\mathbf{x} + \mathbf{c} + A^T\mathbf{y}^*.$$

Also, the Hessian of $L(\mathbf{x}, \mathbf{y}^*)$ is given by

$$\nabla^2_{\mathbf{xx}} L(\mathbf{x}, \mathbf{y}^*) = Q,$$

which is positive semidefinite. Hence, the critical points of $L(\mathbf{x}, \mathbf{y}^*)$ are minimizers. All that remains is to show that we can select a feasible minimizer.

Let $\mathbf{x}_0$ be a minimizer of $L(\mathbf{x}, \mathbf{y}^*)$. Then,

$$g(\mathbf{y}^*) = \inf_{\mathbf{x} \in \mathbb{R}^n} L(\mathbf{x}, \mathbf{y}^*) = L(\mathbf{x}_0, \mathbf{y}^*).$$

Define $\mathcal{H} = \left\{\mathbf{h} \in \mathbb{R}^m \colon A^T\mathbf{h} \in \mathrm{col}(Q)\right\}$. Then, for each $\mathbf{h} \in \mathcal{H}$, we have

$$\mathbf{c}^T + A^T\left(\mathbf{y}^* + \mathbf{h}\right) = \left(\mathbf{c}^T + A^T\mathbf{y}^*\right) + A^T\mathbf{h} \in \mathrm{col}(Q).$$

Hence, by Theorem 4.1, $g(\mathbf{y}^* + \mathbf{h})$ is finite. Moreover,

$$\begin{aligned} g(\mathbf{y}^* + \mathbf{h}) &\leq L(\mathbf{x}_0, \mathbf{y}^* + \mathbf{h}) \\ &= L(\mathbf{x}_0, \mathbf{y}^*) + \mathbf{h}^T\left(A\mathbf{x}_0 - \mathbf{b}\right) \\ &= g(\mathbf{y}^*) + \mathbf{h}^T\left(A\mathbf{x}_0 - \mathbf{b}\right) \end{aligned}$$

Since $\mathbf{y}^*$ maximizes $g(\mathbf{y})$ over $\mathcal{Y}$, it follows that

$$\mathbf{h}^T\left(A\mathbf{x}_0 - \mathbf{b}\right) = 0.$$

Hence,

$$A\mathbf{x}_0 - \mathbf{b} \in \mathcal{H}^\perp = A\mathrm{nul}(Q) = \left\{A\mathbf{z} \colon \mathbf{z} \in \mathrm{nul}(Q)\right\}.$$

Now, let $\mathbf{z}_0 \in \mathrm{nul}(Q)$ such that $A\mathbf{x}_0 - \mathbf{b} = -A\mathbf{z}_0$. Then, define $\mathbf{x}^* = \mathbf{x}_0 + \mathbf{z}_0$. Note that

$$Q\mathbf{x}^* + \mathbf{c} + A^T\mathbf{y}^* = Q\mathbf{x}_0 + \mathbf{c} + A^T\mathbf{y}^* = 0,$$

so $\mathbf{x}^*$ is a critical point (minimizer) of $L(\mathbf{x}, \mathbf{y}^*)$. Moreover,

$$A\mathbf{x}^* = A\mathbf{x}_0 + A\mathbf{z}_0 = \mathbf{b},$$

so $\mathbf{x}^*$ is feasible. □