Carathéodory's Theorem

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February 4, 2026

1 Carathéodory's Theorem

Let $X \subseteq \mathbb{R}^n$. The convex hull of X, denoted $\text{convHull}(X)$, is the smallest convex set that contains X. The convex span of X, denoted $\text{convSpan}(X)$, is the set of all convex combinations of finitely many points from X. Recall that we proved the following proposition in the previous lecture.

Proposition 1. *Let $X \subseteq \mathbb{R}^n$. Then, $\text{convHull}(X) \subseteq \text{convSpan}(X)$.*

In this lecture, we use Carathéodory's theorem to prove that $\text{convSpan}(X) \subseteq \text{convHull}(X)$, which combined with Proposition 1 implies that $\text{convSpan}(X) = \text{convHull}(X)$. To this end, for each $k \in \mathbb{N}$, define $\text{convSpan}_k(X)$ as the set of all convex combinations of k points from X. Note that $\text{convSpan}_1(X) = X$, $\text{convSpan}_2(X)$ is the set of all line segments between two points of X,

$$\text{convSpan}(X) = \bigcup_{k=1}^{\infty} \text{convSpan}_k(X).$$

The $\text{convSpan}_3(X)$ is called an X-triangle. Figure 1 displays all possible X-triangles where X is a collection of 3 extreme points. An important point, related to Carathéodory's Theorem, is that every point in the polytope is contained in a X-triangle.

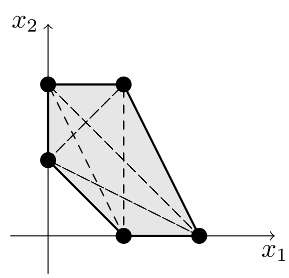

Figure 1: All possible X-triangles, where X is a collection of 3 extreme points.

The following proposition shows that $\text{convSpan}_k(X)$ is a subset of $\text{convHull}(X)$ for each $k \in \mathbb{N}$.

Proposition 2. *Let $X \subseteq \mathbb{R}^n$ and let S denote any convex set that contains X. Then, $\text{convSpan}_k(X) \subseteq S$ for all $k \in \mathbb{N}$.*

Proof. We proceed via induction on k. The base case, when $k = 1$, is clear. Suppose that $\text{convSpan}_k(X) \subseteq S$ for some $k \geq 1$, and let $\mathbf{x} \in \text{convSpan}_{k+1}(X)$. Then, there exists $\mathbf{x}_1, \ldots, \mathbf{x}_{k+1} \in X$ and $c_1, \ldots, c_{k+1} \in \mathbb{R}$ such that

$$\mathbf{x} = c_1\mathbf{x}_1 + \cdots + c_{k+1}\mathbf{x}_{k+1},$$

where $c_i \geq 0$, for all $i \in \{1, \ldots, k+1\}$, and $\sum_{i=1}^{k+1} c_i = 1$. We can assume that $c_1 < 1$; otherwise, $\mathbf{x} = \mathbf{x}_1$ which implies that $\mathbf{x} \in X \subseteq S$. Now, we can rewrite $\mathbf{x}$ as follows

$$\begin{aligned}\mathbf{x} &= c_1\mathbf{x}_1 + (1 - c_1)\left(\frac{c_2}{1 - c_1}\mathbf{x}_2 + \cdots + \frac{c_{k+1}}{1 - c_1}\mathbf{x}_{k+1}\right) \\ &:= c_1\mathbf{x}_1 + (1 - c_1)\gamma.\end{aligned}$$

Note that $\frac{c_i}{1-c_1} \geq 0$ for all $i \in \{2, \dots, k+1\}$. Moreover,

$$\sum_{i=2}^{k+1} \frac{c_i}{1-c_1} = \frac{1}{1-c_1} \sum_{i=2}^{k+1} c_i = \frac{1-c_1}{1-c_1} = 1.$$

Thus, $\gamma \in \text{convSpan}_k(X)$, so the induction hypothesis gives $\gamma \in S$. Therefore, $\mathbf{x}$ lies along a line segment between two points in S. Since S is a convex set, it follows that $\mathbf{x} \in S$. □

In 1911, Constantin Carathéodory proved that every point in the convex span of X can be written as a convex combination of at most $n+1$ points from X. We prove this result in Theorem 3.

Theorem 3 (Constantin Carathéodory)**.** *Let* $X \subseteq \mathbb{R}^n$. *Then,*

$$\text{convSpan}(X) = \text{convSpan}_{n+1}(X).$$

Proof. By definition, $\text{convSpan}_{n+1}(X) \subseteq \text{convSpan}(X)$. Hence, we only need to show the reverse containment. To this end, let $\gamma \in \text{convSpan}(X)$. Then,

$$\gamma = t_1\mathbf{x}_1 + \cdots + t_k\mathbf{x}_k,$$

where $k \in \mathbb{N}$, $\mathbf{x}_1, \dots, \mathbf{x}_k \in X$, $t_j \geq 0$ for each $j \in \{1, \dots, k\}$, and $\sum_{j=1}^{k} t_j = 1$.

We may assume that $k \geq n+1$; otherwise, $\gamma \in \text{convSpan}_{n+1}(X)$ and we are done. Now, denote by x_{ij} the ith entry of $\mathbf{x}_j$. Then the coefficients t_j correspond to a feasible solution of the following LP.

$$\begin{aligned} \text{maximize} \quad & z = t_1 + \cdots + t_k \\ \text{subject to} \quad & t_1x_{i1} + \cdots + t_kx_{ik} = \gamma_i,\ 1 \leq i \leq n, \\ & t_1 + \cdots + t_k = 1, \\ & t_j \geq 0,\ 1 \leq j \leq k \end{aligned}$$

Note that, using row operations, we can reduce the above LP to a dictionary form, where each constraint corresponds to at most one basic variable t_j. More importantly, since the above LP is feasible its corresponding tableau must have a feasible basic solution. Since there are at most $(n+1)$ basic variables, it follows that any basic solution has at most $(n+1)$ non-zero entries. Therefore, γ can be written as a convex combination of at most $(n+1)$ points from X. □

Theorem 3 combined with Propositions 1 and 2 implies the following result.

Corollary 4. *Let* $X \subseteq \mathbb{R}^n$. *Then,* $\text{convSpan}(X) = \text{convHull}(X)$.

1.1 Class Exercises

Let $X = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \mathbf{x}_3 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

Also, let $\gamma = \sum_{j=1}^{4} \frac{1}{4}\mathbf{x}_i$.

I. Use Desmos to plot the set X and γ.

II. On the same Desmos plot, plot the line segment from $\mathbf{x}_2$ to $\mathbf{x}_4$. Use this plot to determine γ as a convex combination of $\mathbf{x}_2$ and $\mathbf{x}_4$.

III. Use the proof technique from Theorem 3 to determine γ as a convex combination of at most three points from X.