Topology of the Reals

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1 Topology of the Reals

We've seen that the real numbers $\mathbb{R}$ are a complete ordered field. Moreover, this field has a well-defined metric which induces a topology on $\mathbb{R}$, that is, a notion of nearness. More specifically, let $x \in \mathbb{R}$ and $\epsilon \in \mathbb{R}_{>0}$. Then, the *epsilon-neighborhood* of x is defined by

$$N(x;\epsilon) = \{y \in \mathbb{R} \colon \ |x - y| < \epsilon\}$$

and the *deleted epsilon-neghborhood* of x is defined by

$$N^*(x;\epsilon) = \{y \in \mathbb{R} \colon 0 < |x - y| < \epsilon\}.$$

The neighborhoods motivate topological definitions of points in relation to a subset of real numbers. In particular, let $S \subseteq \mathbb{R}$.

(a) A point $x \in S$ is a *interior point* of S if there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $N(x;\epsilon) \subseteq S$.

(b) A point $x \in \mathbb{R}$ is a *boundary point* of S if for all $\epsilon \in \mathbb{R}_{>0}$, $N(x;\epsilon) \cap S \neq \emptyset$ and $N(x;\epsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$.

(c) A point $x \in \mathbb{R}$ is an *accumulation point* of S if for all $\epsilon \in \mathbb{R}_{>0}$, $N^*(x;\epsilon) \cap S \neq \emptyset$.

(d) A point $x \in S$ is an *isolated point* of S if there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $N^*(x;\epsilon) \cap S = \emptyset$.

Note that you can move a small distance away from an interior point and remain in the set S. With a boundary point, if you move any distance you can end up at point in S or a point not in S. For an accumulation point, no matter how close you get to the point there will be other points from the set S. For isolated points, if you get to close then there are no other points of S.

Given a subset $S \subseteq \mathbb{R}$, the set of interior points is denoted by $\operatorname{int}(S)$ and the set of boundary points is denoted by $\operatorname{bd}(S)$. The set S is said to be *closed* if it contains all of its boundary points, that is, $\operatorname{bd}(S) \subseteq S$. If the set S contains none of its boundary points, that is, $\operatorname{bd}(S) \subseteq (\mathbb{R} \setminus S)$, then the set is *open*. For example, the set $S = \{x \in \mathbb{R} \colon 0 < x < 2\}$ has boundary points $\operatorname{bd}(S) = \{0, 2\}$. Since $\operatorname{bd}(S) \subseteq (\mathbb{R} \setminus S)$, it follows that S is open. Note that the state of being open or closed is not mutually exclusive; for example, the empty set $\emptyset$ is both open and closed. Moreover, a set may be neither open nor closed; for example, the set $(0, 4]$ is neither open nor closed.

The following result provides a characterization of open and closed subsets.

Theorem 1.1. *Let $S \subseteq \mathbb{R}$. Then,*

(a) S is open if and only if $S = \operatorname{int}(S)$,

(b) S is closed if and only if its complement $\overline{S} = \mathbb{R} \setminus S$ is open.

Proof. (a) If S is empty, then the result is trivial since $S = \operatorname{int}(S) = \emptyset$. Hence, we assume that S is non-empty. Suppose S is open, then S contains none of its boundary points. Let $x \in S$. Then, x is not a boundary point of S. Therefore, there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $N(x;\epsilon) \cap (\mathbb{R} \setminus S) = \emptyset$, that is, $N(x;\epsilon) \subseteq S$. Thus, x is an interior point of S, so we have established that $S \subseteq \operatorname{int}(S)$. Since $\operatorname{int}(S) \subseteq S$ by definition, it follows that $S = \operatorname{int}(S)$.

Conversely, suppose that $S = \operatorname{int}(S)$. Then, $S \subseteq \operatorname{int}(S)$. Since no interior point can be a boundary point, it follows that S contains none of its boundary points. Therefore, S is open.

(b) Suppose that S is closed. Then, $\mathrm{bd}\,(S) \subseteq S$. Let $x \in \overline{S}$. Then, x is not a boundary point of S. Hence, there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $N(x;\epsilon) \cap S = \emptyset$ or $N(x;\epsilon) \cap (\mathbb{R} \setminus S) = \emptyset$. Since $x \in \overline{S}$, it follows that $N(x;\epsilon) \cap S = \emptyset$, that is, $N(x;\epsilon) \subseteq \overline{S}$. Therefore, x is an interior point of $\overline{S}$, so we have established that $\overline{S}$ is open.

Conversely, suppose that $\overline{S}$ is open. Then, $\overline{S} \subseteq \mathrm{int}\left(\overline{S}\right)$. For the sake of contradiction, suppose that $x \in \mathrm{bd}\,(S)$ and $x \notin S$, that is, $x \in \overline{S}$. Since $\overline{S}$ is open, x must be an interior point of $\overline{S}$. Thus, there exists a $\epsilon \in \mathbb{R}_{>0}$ such that $N(x;\epsilon) \subseteq \overline{S}$, which contradicts $N(x;\epsilon) \cap S \neq \emptyset$. Therefore, $\mathrm{bd}\,(S) \subseteq S$, so S is closed.

□

We can further characterize closed sets using accumulation points. First, we define the *closure* of S to be

$$\mathrm{cl}\,(S) = S \cup S'$$

where S' denotes the set of accumulation points of S. In terms of neighborhoods, a point x is in the closure of S if and only if every neighborhood of x intersects S. The following result establishes properties of the closure.

Theorem 1.2. *Let $S \subseteq \mathbb{R}$. Then,*

(a) $\mathrm{cl}\,(S)$ *is a closed set,*

(b) $\mathrm{cl}\,(S) = S \cup \mathrm{bd}\,(S)$.

Proof. (a) By Theorem 1.1, it suffices to show that the complement of $\mathrm{cl}\,(S)$ is open. To this end, note that

$$\begin{aligned}\overline{\mathrm{cl}\,(S)} &= \{x \in \mathbb{R} \colon x \notin S \wedge x \notin S'\} \\ &= \left\{x \in \mathbb{R} \colon \exists \epsilon \in \mathbb{R}_{>0} \ni N(x;\epsilon) \subseteq \overline{S}\right\}.\end{aligned}$$

Suppose that $x \in \overline{\mathrm{cl}\,(S)}$. Then, there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $N(x;\epsilon) \subseteq \overline{S}$. In fact, we will show that $N(x;\epsilon) \subseteq \overline{\mathrm{cl}\,(S)}$, which implies that $\mathrm{cl}\,(S)$ is closed by Theorem 1.1. To this end, let $y \in N(x;\epsilon)$. Then, there exists an $\epsilon' > 0$ such that $N(y;\epsilon') \subsetneq N(x;\epsilon) \subseteq \overline{S}$. Therefore, every element in $N(x;\epsilon)$ is in $\overline{\mathrm{cl}\,(S)}$.

(b) Suppose that $x \in \mathrm{cl}\,(S) = S \cup S'$. Then, $x \in S$ or $x \in \mathrm{cl}\,(S)$. If $x \in S$, then $x \in S \subseteq S \cup \mathrm{bd}\,(S)$. Suppose that $x \notin S$ and $x \in S'$. Then, for all $\epsilon \in \mathbb{R}_{>0}$, $N(x;\epsilon) \cap S \neq \emptyset$ and $N(x;\epsilon) \cap (\mathbb{R} \setminus S) \neq \emptyset$. Therefore, $x \in \mathrm{bd}\,(S) \subseteq S \cup \mathrm{bd}\,(S)$.

Suppose that $x \in S \cup \mathrm{bd}\,(S)$. If $x \in S$, then $x \in S \subseteq S \cup S' = \mathrm{cl}\,(S)$. Suppose that $x \notin S$ and $x \in \mathrm{bd}\,(S)$. Then, for all $\epsilon \in \mathbb{R}_{>0}$, $N^*(x;\epsilon) \cap S \neq \emptyset$. Therefore, $x \in S' \subseteq S \cup S' = \mathrm{cl}\,(S)$.

□

The following result uses the closure to establish further characterizations of closed sets.

Theorem 1.3. *Let $S \subseteq \mathbb{R}$. Then,*

(a) S is closed if and only if S contains all of its accumulation points.

(b) S is closed if and only if $S = \mathrm{cl}\,(S)$.