

Subsequences

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1 Subsequences

Let $s: \mathbb{N} \rightarrow \mathbb{R}$ be a sequence. Then, for any strictly increasing sequence $\sigma: \mathbb{N} \rightarrow \mathbb{N}$, the composition $s \circ \sigma$ is a *subsequence* of s . Furthermore, the terms of the subsequence are denoted by

$$s \circ \sigma = (s_{\sigma_1}, s_{\sigma_2}, s_{\sigma_3}, \dots).$$

For example, let $s = (1/n)_{n=1}^{\infty}$. Then, $\sigma = (2k)_{k=1}^{\infty}$ induces the subsequence

$$s \circ \sigma = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots \right).$$

Similarly, $\tau = (2k-1)_{k=1}^{\infty}$ induces the subsequence

$$s \circ \tau = \left(\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \dots \right).$$

Subsequences can be used to describe properties of convergent sequences; in particular, a convergent sequence has every subsequence converging to the same value.

Theorem 1.1. *Let $s: \mathbb{N} \rightarrow \mathbb{R}$ be a convergent sequence with limiting value $L \in \mathbb{R}$. Then, every subsequence of s converges to L .*

Proof. Let $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing sequence. Then, $\sigma_k \geq k$ for all $k \in \mathbb{N}$.

Now, let $\epsilon \in \mathbb{R}_{>0}$. Since $\lim_{n \rightarrow \infty} s_n = L$, there exists a $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow |s_n - L| < \epsilon.$$

Therefore, $k \geq N$ implies that $\sigma_k \geq k \geq N$, so we have

$$|s_{\sigma_k} - L| < \epsilon.$$

Thus, $\lim_{k \rightarrow \infty} s_{\sigma_k} = L$. □

Theorem 1.1 is useful for showing when a sequence diverges. For example, consider the sequence

$$((-1)^{n+1})_{n=1}^{\infty} = (1, -1, 1, -1, \dots).$$

This sequence has subsequences $(1, 1, 1, 1, \dots)$ and $(-1, -1, -1, -1, \dots)$. Since these subsequences do not converge to the same value it follows that the sequence diverges. As another example, consider the sequence

$$(n)_{n=1}^{\infty} = (1, 2, 3, 4, \dots).$$

This sequence has no convergent subsequences and therefore must diverge.

The following result shows that if a sequence is bounded, then it has at least one convergent subsequence.

Theorem 1.2. *Let $s: \mathbb{N} \rightarrow \mathbb{R}$ be a bounded sequence. Then, there exists a subsequence of s that converges.*

Proof. Let $\text{rng}(s) = \{s_n : n \in \mathbb{N}\}$ denote the range of s . If $\text{rng}(s)$ is finite, then there is some value $L \in \text{rng}(s)$ that is equal to s_n for infinitely many values of n . Thus, there exists $\sigma_1 < \sigma_2 < \sigma_3 \cdots$ such that $s_{\sigma_k} = L$ for all $k \in \mathbb{N}$. Therefore, the subsequence $s \circ \sigma$ is convergent.

If $\text{rng}(s)$ is infinite, then the Bolzano-Weiestrass theorem states that $\text{rng}(s)$ has an accumulation point, which we denote by $L \in \mathbb{R}$. Since L is an accumulation point, for each $k \in \mathbb{N}$ there are infinitely many points in $N(L; 1/k) \cap \text{rng}(s)$. In particular, there exists a $\sigma_1 \in \mathbb{N}$ such that $s_{\sigma_1} \in N(L; 1)$. In addition, there exists a $\sigma_2 \in \mathbb{N}$ such that $\sigma_2 > \sigma_1$ and $s_{\sigma_2} \in N(L; 1/2)$. In general, for $k \geq 3$, we can select a $\sigma_k \in \mathbb{N}$ such that $\sigma_k > \sigma_{k-1}$ and $s_{\sigma_k} \in N(L; 1/k)$. Therefore, for each $k \in \mathbb{N}$, we have

$$|s_{\sigma_k} - L| < \frac{1}{k},$$

which implies that the subsequence $(s_{\sigma_k})_{k=1}^{\infty}$ converges to L . □