Set Theory

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August 27, 2025

1 Sets

A *set* is a repetition-free, unordered collection of objects. An object that belongs to a set is called an *element* of that set. We denote membership to a set A by $x \in A$. If x is not a member of A, we write $x \notin A$. The number of elements in A, denoted by $|A|$, is called the *cardinality* of A. A set is *finite* if its cardinality is an integer; otherwise, the set is *infinite*. The *empty set* is the set with no members, which we denote by $\emptyset$.

Let A and B be sets. We say that A is a *subset* of B, denoted by $A \subseteq B$, if every element of A is also an element of B. If there is an element of B that is not in A, then A is a *proper subset* of B, which we denote by $A \subsetneq B$. We say that A is *equal* to B, denoted by $A = B$, if $A \subseteq B$ and $B \subseteq A$. To prove that $A \subseteq B$, we must prove the following implication $x \in A \Rightarrow x \in B$. To prove that $A = B$, we must prove both implications $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$.

In Proposition 1.1 we give an example of proving set equality. Note that the set of integers is denoted by $\mathbb{Z}$. Given $n \in \mathbb{Z}$, we say that n is *even* if $n = 2k$ for some $k \in \mathbb{Z}$, and n is *odd* if $n = 2k + 1$ for some $k \in \mathbb{Z}$.

Proposition 1.1. *Let* $A = \{n \in \mathbb{Z} \colon n \text{ is even}\}$ *and* $B = \{n \in \mathbb{Z} \colon n = a + b, \text{ where } a \text{ and } b \text{ are odd}\}$. *Then,* $A = B$.

Proof. First, we show that $A \subseteq B$. To this end, let $n \in A$. Then, $n = 2k = k + k$ for some $k \in \mathbb{Z}$. If k is odd, then it is clear that $n \in B$. Otherwise, $a = k + 1$ and $b = k - 1$ are both odd and $n = a + b$, which implies that $n \in B$.

Second, we show that $B \subseteq A$. To this end, let $n \in B$. Then, $n = a + b$ for some odd integers a and b. Since a and b are both odd, there exist integers k and k' such that $a = 2k + 1$ and $b = 2k' + 1$. Therefore, $n = a + b = 2(k + k' + 1)$ is even. □

2 Set Operations

Let A and B be sets. The *union* of A and B, denoted $A \cup B$, is the set of all elements in A or B. The *intersection* of A and B, denoted $A \cap B$, is the set of all elements in both A and B. We can write the union and intersection as follows

$$A \cup B = \{x \colon x \in A \wedge x \in B\}, \; A \cap B = \{x \colon x \in A \vee x \in B\}.$$

The union and intersection have various algebraic properties as outlined in Proposition 2.1.

Proposition 2.1. *Let* A*,* B*, and* C *denote sets. Then,*

(a) $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$

(b) $A \cup (B \cup C) = (A \cup B) \cup C$ *and* $A \cap (B \cap C) = (A \cap B) \cap C$

(c) $A \cup \emptyset = A$ *and* $A \cap \emptyset = \emptyset$

(d) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *and* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. We will prove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

the remaining properties will be left as exercises.

Suppose that $x \in A \cup (B \cap C)$. Then, $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, which implies that $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B$ and $x \in C$. Therefore, $x \in A \cup B$ and $x \in A \cup C$, which implies that $x \in (A \cup B) \cap (A \cup C)$.

Conversely, suppose that $x \in (A \cup B) \cap (A \cup C)$. Then, $x \in (A \cup B)$ and $x \in (A \cup C)$, which implies that $x \in A$ or $x \in B \cap C$. Therefore, $x \in A \cup (B \cap C)$. □

The *set difference*, denoted $A - B$ or $A \setminus B$, is the set of all elements in A that are not in B, that is,

$$A - B = \{x \in A \colon x \notin B\}.$$

We say that A and B are *disjoint* if $A \cap B = \emptyset$. In this case, we have $A - B = A$ and $B - A = B$. In Theorem 2.2, we state DeMorgan's law for sets.

Theorem 2.2 (DeMorgan's Law)**.** *Let A, B, and C be sets. Then,*

$$A - (B \cup C) = (A - B) \cap (A - C)$$

and

$$A - (B \cap C) = (A - B) \cup (A - C).$$

3 Relations

The *ordered pair* (a, b) is an ordered set of two elements where

$$(a, b) = (c, d) \Leftrightarrow a = c \wedge b = d.$$

We can use sets to define an ordered pair as follows

$$(a, b) = \{a, \{a, b\}\}.$$

Let A and B be sets. The *Cartesian product* of A and B, denoted $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$, that is,

$$A \times B = \{(a, b) \colon a \in A \wedge b \in B\}$$

A *relation* bewteen A and B is any subset $R \subseteq A \times B$. We say that $a \in A$ and $b \in B$ are related by R if $(a, b) \in R$. We say that the relation $R \subseteq A \times A$ is an *equivalence relation* if the following properties holds

(a) Reflexive: for all $x \in A$, $(x, x) \in R$

(b) Symmetric: if $(x, y) \in R$, then $(y, x) \in R$

(c) Transitive: if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.

Given an equivalence relation R on the set A, it is natural to group the elements that are related to a particular element. More precisely, we define the *equivalence class*, with respect to R, of $x \in A$ by

$$E_x = \{y \in A \colon (x, y) \in R\}.$$

The equivalence classes form a partition of A. In general, a *partition* of A is a set $\mathcal{P}$ of non-empty subsets of A such that

(a) for all $x \in A$, there exists a $P \in \mathcal{P}$ such that $x \in P$,

(b) for all $P, P' \in \mathcal{P}$, either $P = P'$ or $P \cap P' = \emptyset$.

Note that any member of $\mathcal{P}$ is called a *piece of the partition*. Moreover, given a partition $\mathcal{P}$ of the set A, we can define an equivalence relation R as follows: $(x, y) \in R$ if and only if x and y are in the same piece of the partition $\mathcal{P}$.

4 Functions

Let A and B be sets. A *funciton* between A and B is a non-empty relation $f \subseteq A \times B$ such that if $(a, b) \in f$ and $(a, b') \in f$ then $b = b'$. The *domain* and *range* of f, denoted $\operatorname{dom}(f)$ and $\operatorname{rng}(f)$, respectively, is defined as follows

$$\operatorname{dom}(f) = \{a \in A \colon \exists b \in B \ \ni \ (a, b) \in f\}$$
$$\operatorname{rng}(f) = \{b \in B \colon \exists a \in A \ \ni \ (a, b) \in f\}$$

The set B is referred to as the codomain of f. If the domain of f contains all of A, we say that f is a function from A into B and we write $f \colon A \to B$.

A function f is *surjective* if $\operatorname{rng}(f) = B$ and is *injective* if $f(a) = f(a')$ only when $a = a'$. Moreover, f is *bijective* if it is injective and surjective. For example, let $f(x) = x^2$. Then, $f \colon \mathbb{R} \to \mathbb{R}$ is neither injective nor surjective; $f \colon \mathbb{R} \to [0, \infty)$ is surjective but not injective; $f \colon [0, \infty) \to [0, \infty)$ is bijective.

Let A, B, and C be sets. Given functions $f \colon A \to B$ and $g \colon B \to C$, the *composition* is defined by

$$g \circ f = \{(a, c) \in A \times C \colon \exists b \in B \ \ni \ (a, b) \in f \wedge (b, c) \in g\}.$$

The identity function on A is defined by

$$i_A = \{(a, a) \colon a \in A\}.$$

Suppose that $f \colon A \to B$ is a bijection. Then, the *inverse function* is defined by

$$f^{-1} = \{(b, a) \colon (a, b) \in f\}.$$

Theorem 4.1. *Let A and B be sets and let $f \colon A \to B$ be a bijection. Then, f^{-1} is a bijective function from B onto A. Moreover, the following compositions hold*

$$f^{-1} \circ f = i_a \text{ and } f \circ f^{-1} = i_B.$$