Series

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1 Preliminaries

Recall that a sequence is a function of the form $s\colon \mathbb{N} \to \mathbb{R}$. We say that s converges if there exists an $L \in \mathbb{R}$ such that for all $\epsilon \in \mathbb{R}_{>0}$ there is a $N \in \mathbb{N}$ such that $|s_n - L| < \epsilon$ whenever $n \geq N$. In this case, we write $\lim_{n\to\infty} s_n = L$; if no such L exists, then we say that s diverges. For real sequences, we can say that s converges if and only if s is Cauchy, where a sequence s is Cauchy if for all $\epsilon \in \mathbb{R}$ there is a $N \in \mathbb{N}$ such that $|s_n - s_m| < \epsilon$ whenever $n, m \geq N$.

2 Series

Given a sequence $a\colon \mathbb{N} \to \mathbb{R}$, we define the sequence of *partial sums* by

$$s_n = \sum_{k=1}^{n} a_k.$$

The *series* is defined as the limit of the sequence of partial sums, that is,

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} s_n$$

We say that the series converges if the sequence of partial sums converges; otherwise, we say that the series diverges.

For example, consider the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$, which has a sequence of partial sums

$$\begin{aligned} s_1 &= \frac{1}{2} \\ s_2 &= \frac{1}{2} + \frac{1}{6} = \frac{2}{3} \\ s_3 &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4} \\ &\vdots \\ s_n &= \frac{n}{n+1}. \end{aligned}$$

Therefore, $\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to\infty} \frac{n}{n+1} = 1$.

As another example, consider the series $\sum_{k=0}^{\infty} \frac{1}{2^k}$, which has a sequence of partial sums

$$\begin{aligned} s_0 &= 1 \\ s_1 &= 1 + \frac{1}{2} = \frac{3}{2} \\ s_2 &= 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4} \\ s_3 &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{15}{8} \\ &\vdots \\ s_n &= \frac{2^{n+1}-1}{2^n}. \end{aligned}$$

Therefore, $\sum_{k=0}^{\infty} \frac{1}{2^k} = \lim_{n\to\infty} \frac{2^{n+1}-1}{2^n} = 2$.

Since a real sequence converges if and only if that sequence is Cauchy, we have the following characterization of convergent series.

Theorem 2.1. *Let $a\colon \mathbb{N} \to \mathbb{R}$. Then, the series $\sum_{k=1}^{\infty}$ converges if and only if for all $\epsilon \in \mathbb{R}_{>0}$, there is a $N \in \mathbb{N}$ such that*

$$|a_{m+1} + \cdots + a_n| < \epsilon.$$

Proof. For each $n \in \mathbb{N}$, let $s_n = \sum_{k=1}^{n} a_k$ denote the nth partial sum. Suppose that the series converges. Then, the sequence of partial sums converges and is therefore Cauchy. Let $\epsilon \in \mathbb{R}_{>0}$, then there is a $N \in \mathbb{N}$ such that $|s_n - s_m| < \epsilon$ whenever $n, m \geq N$. Therefore, whenever $n \geq m \geq N$, we have $n \geq N$ and $m \geq N$, thus $|s_n - s_m| < \epsilon$. Since $s_n - s_m = a_{m+1} + \cdots + a_n$, it follows that

$$|a_m + a_{m+1} + \cdots + a_n| < \epsilon,$$

whenever $n \geq m \geq N$.

Conversely, suppose that for all $\epsilon \in \mathbb{R}_{>0}$, there is a $N \in \mathbb{N}$ such that

$$|a_{m+1} + \cdots + a_n| < \epsilon,$$

whenever $n \geq m \geq N$. Then, for all $n, m \geq N$, it follows that $|s_n - s_m| < \epsilon$. Therefore, the sequence of partial sums is Cauchy and therefore converges; hence, there series converges. □

As a corollary of Theorem 2.1, we see that convergent series have terms that converge to zero.

Corollary 2.2. *Let $a\colon \mathbb{N} \to \mathbb{R}$. If the series $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.*

Proof. Suppose that the series $\sum_{k=1}^{\infty} a_k$ converges. By Theorem 2.1, for any $\epsilon \in \mathbb{R}_{>0}$ there is a $N \in \mathbb{N}$ such that

$$|s_n - s_m| = |a_{m+1} + \cdots + a_n| < \epsilon,$$

whenever $n \geq m \geq N$.

By setting $n = m + 1$, we see that $|a_{m+1}| < \epsilon$ for all $m \geq N$. Since this holds for any $\epsilon \in \mathbb{R}_{>0}$, it follows that $\lim_{k\to\infty} a_k = 0$. □

3 Geometric and Telescoping Series

A *geometric series* is a series that can be written in the form

$$\sum_{k=0}^{\infty} ar^k,$$

where $a, r \in \mathbb{R} \setminus \{0\}$. The corresponding sequence of partial sums

$$s_n = \sum_{k=0}^{n} ar^k = a + ar + ar^2 + \cdots + ar^n$$

clearly diverges if $|r| \geq 1$. Note that $s_n - rs_n = a - ar^{n+1}$. Therefore,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a - ar^{n+1}}{1-r} = \frac{a}{1-r},$$

provided that $|r| < 1$.

As an example, consider the series

$$\sum_{k=0}^{\infty} \frac{3}{5^k} = \frac{3}{1-1/5} = \frac{15}{4}.$$

As another example, consider the series

$$\begin{aligned}\sum_{k=2}^{\infty} \frac{3}{5^k} &= \sum_{k=0}^{\infty} \frac{3}{5^{k+2}} \\ &= \sum_{k=0}^{\infty} \frac{3}{5^2}\frac{1}{5^k} \\ &= \frac{3/25}{1-1/5} = \frac{3}{20}\end{aligned}$$

A *telescoping series* is a series that can be written in the form

$$\sum_{k=1}^{\infty} (a_k - a_{k-1}) \quad \text{or} \quad \sum_{k=1}^{\infty} (a_{k-1} - a_k),$$

so the partial sum can be written as

$$s_n = \sum_{k=1}^{n} (a_k - a_{k-1}) = a_n - a_0 \quad \text{or} \quad s_n = \sum_{k=1}^{\infty} (a_{k-1} - a_k) = a_0 - a_n.$$

In either case, the telescoping series converges if and only if $\lim_{n\to\infty} a_n$ exists and is finite.

As an example, consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

As another example, consider the series

$$\begin{aligned}\sum_{k=2}^{\infty} \frac{1}{k^2-1} &= \sum_{k=2}^{\infty} \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) \\ &= \frac{1}{2}\sum_{k=2}^{\infty} \left(\frac{1}{k-1} - \frac{1}{k} + \frac{1}{k} - \frac{1}{k+1}\right) \\ &= \frac{1}{2}\lim_{n\to\infty}\left(1 - \frac{1}{n}\right) + \frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{n+1}\right) \\ &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.\end{aligned}$$

4 Harmonic Series

By Corollary 2.2, all convergent series have terms that converge to zero. While this is necessary, it is not sufficient for convergence. Our first example of a divergent series that has terms that converge to zero is the *harmonic series*, which is defined as $\sum_{k=1}^{\infty} \frac{1}{k}$. Note that the sequence of partial sums satisfy

$$\begin{aligned}
s_2 &= 1 + \frac{1}{2} > \frac{1}{2} + \frac{1}{2} = \frac{2}{2} \\
s_4 &= s_2 + \frac{1}{3} + \frac{1}{4} > s_2 + 2\frac{1}{4} > \frac{3}{2} \\
s_8 &= s_4 + \frac{1}{5} + \cdots + \frac{1}{8} > s_4 + 4\frac{1}{8} > \frac{4}{2} \\
s_{16} &= s_8 + \frac{1}{9} + \cdots + \frac{1}{16} > s_8 + 8\frac{1}{16} > \frac{5}{2} \\
&\vdots \\
s_{2^n} &> \frac{n+1}{2}.
\end{aligned}$$

Therefore, the sequence of partial sums diverges and so the harmonic series also diverges.