Sequences

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October 1, 2025

1 Convergence

A *sequence* is a function $s\colon \mathbb{N} \to \mathbb{R}$. We denote the nth term of the sequence by $s(n) = s_n$; we may also denote the sequence by

$$(s_n)_{n=1}^{\infty} = (s_1, s_2, s_3, s_4, \ldots).$$

For example, the sequence

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots$$

can be written as $\left(\frac{1}{n}\right)_{n=1}^{\infty}$. As another example, the sequence

$$\frac{1}{3},\ \frac{2}{5},\ \frac{3}{7},\ \frac{4}{9},\ \ldots$$

can be written as $\left(\frac{n}{2n+1}\right)_{n=1}^{\infty}$.

A sequence $(s_n)_{n=1}^{\infty}$ *converges* to $L \in \mathbb{R}$ if for all $\epsilon \in \mathbb{R}_{>0}$, there is a $N \in \mathbb{N}$ such that $|s_n - L| < \epsilon$ whenever $n \geq N$. If the sequence $(s_n)_{n=1}^{\infty}$ converges to $L \in \mathbb{R}$, we write $\lim_{n\to\infty} s_n = L$; if $(s_n)_{n=1}^{\infty}$ does not converge to any $L \in \mathbb{R}$, then we say that the sequence diverges. For example, the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ converges to 0. Indeed, let $\epsilon \in \mathbb{R}_{>0}$ and let N be an integer greater than $1/\epsilon$. Then,

$$\begin{aligned} n \geq N \Rightarrow \left|\frac{1}{n} - 0\right| &= \frac{1}{n} \\ &\leq \frac{1}{N} < \epsilon. \end{aligned}$$

Note that the existence of an integer greater than $1/\epsilon$ is guaranteed by the Archimedean property. As another example, the sequence $\left(\frac{n}{2n+1}\right)_{n=1}^{\infty}$ converges to $1/2$. Indeed, let $\epsilon \in \mathbb{R}_{>0}$ and let N be an integer greater than $\frac{1}{4\epsilon}$. Then,

$$\begin{aligned} n \geq N \Rightarrow \left|\frac{n}{2n+1} - \frac{1}{2}\right| &= \frac{1}{4n+2} \\ &\leq \frac{1}{4N+2} < \frac{\epsilon}{1+2\epsilon} < \epsilon. \end{aligned}$$

2 Limit Theorems

The following results states that if a sequence converges then its limiting value is unique.

Proposition 2.1. *Let $s\colon \mathbb{N} \to \mathbb{R}$. If the sequence s converges, then its limit is unique.*

Proof. Suppose that s converges to both $L_1, L_2 \in \mathbb{R}$. Now, let $\epsilon \in \mathbb{R}_{>0}$ Then, exists $N_1, N_2 \in \mathbb{N}$ such that

$$n \geq N_1 \Rightarrow |s_n - L_1| < \epsilon/2$$

and

$$n \geq N_2 \Rightarrow |s_n - L_2| < \epsilon/2.$$

Therefore, $N = \max\{N_1, N_2\}$ satisfies

$$|L_1 - L_2| \leq |L_1 - s_N| + |s_N - L_2| < \epsilon,$$

which implies that $L_1 = L_2$. □

The following result states that every convergent sequence is bounded.

Proposition 2.2. *Every convergent sequence is bounded.*

Proof. Let $s\colon \mathbb{N} \to \mathbb{R}$ be convergent. Also, let $L = \lim_{n\to\infty} s_n$. Then, there exists a $N \in \mathbb{N}$ such that $|s_n - L| < 1$ for all $n \geq N$. Therefore, $|s_n| < 1 + L$ for all $n \geq N$. Define,

$$M = \max\left\{|s_1|, |s_2|, \ldots, |s_{N-1}|, L + 1\right\}.$$

Then, $|s_n| \leq M$ for all $n \in \mathbb{N}$, so the sequence is bounded. □

The following theorem establishes algebraic properties of the limit.

Theorem 2.3. *Let $s\colon \mathbb{N} \to \mathbb{R}$ and $t\colon \mathbb{N} \to \mathbb{R}$ be convergent with limits L and L', respectively. Then, the following equations hold*

(a) $\lim_{n\to\infty}(s_n + t_n) = L + L'$,

(b) $\lim_{n\to\infty}(k \cdot s_n) = kL$, *for any* $k \in \mathbb{R}$,

(c) $\lim_{n\to\infty}(s_n \cdot t_n) = L \cdot L'$,

(d) $\lim_{n\to\infty}\left(\frac{s_n}{t_n}\right) = \frac{L}{L'}$, *provided that* $L' \neq 0$.