

Sequences Definitions Quiz

Math 312: Concepts of Real Analysis

Answer Key

1. Topological definitions.

Let $S \subseteq \mathbb{R}$.

- (a) Define an interior point and a boundary point of S .
- (b) Define an accumulation point and an isolated point of S .
- (c) Define an open set and a closed set in $\mathbb{R}$.

Solution.

- (a) A point $x \in S$ is interior if some $r > 0$ satisfies $N(x; r) \subseteq S$. A point $x \in \mathbb{R}$ is a boundary point if, for every $r > 0$, both $N(x; r) \cap S$ and $N(x; r) \cap S^c$ are nonempty.
- (b) A point $x \in \mathbb{R}$ is an accumulation point if, for every $r > 0$, there exists $s \in S$ with $0 < |s - x| < r$. A point $x \in S$ is isolated if some $r > 0$ satisfies $N(x; r) \cap S = \{x\}$.
- (c) A set is open if every point of it is interior. A set is closed if it contains every accumulation point of the set (equivalently, its complement is open).

2. Compactness and covers.

Let $K \subseteq \mathbb{R}$.

- (a) Define an open cover of K and a finite subcover of that cover. Make clear which sets may be selected.
- (b) Define compactness using open covers and finite subcovers.
- (c) State the Heine–Borel characterization of compact subsets of $\mathbb{R}$.

Solution.

- (a) An open cover is a collection $\mathcal{U}$ of open subsets of $\mathbb{R}$ satisfying $K \subseteq \bigcup_{U \in \mathcal{U}} U$. A finite subcover is a finite subcollection $\mathcal{V} \subseteq \mathcal{U}$ satisfying $K \subseteq \bigcup_{U \in \mathcal{V}} U$. Its sets must come from the original cover.
- (b) The set K is compact if every open cover of K has a finite subcover.
- (c) A set $K \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.

3. Convergence, Cauchy sequences, and monotonicity.

Let (s_n) be a real sequence and $L \in \mathbb{R}$.

- (a) State the definition of $\lim_{n \rightarrow \infty} s_n = L$.
- (b) Negate the definition in part (a): state precisely what it means that (s_n) does not converge to this specified L .
- (c) State the definition of a Cauchy sequence.
- (d) State the definitions of an increasing (nondecreasing) sequence and a decreasing (nonincreasing) sequence.
- (e) State the Cauchy convergence criterion for real sequences.

Solution.

- (a) For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that, for every $n \geq N$, $|s_n - L| < \epsilon$.
- (b) There exists $\epsilon_0 > 0$ such that, for every $N \in \mathbb{N}$, there exists $n \geq N$ with $|s_n - L| \geq \epsilon_0$. Failure to converge to one specified L does not by itself mean divergence.
- (c) A sequence (s_n) is Cauchy if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|s_m - s_n| < \epsilon$ for all $m, n \geq N$.
- (d) It is increasing (nondecreasing) if $s_n \leq s_{n+1}$ for every $n \in \mathbb{N}$, and decreasing (nonincreasing) if $s_n \geq s_{n+1}$ for every $n \in \mathbb{N}$.
- (e) A real sequence converges to a real number if and only if it is Cauchy.