

Exam 2 Review Worksheet

MATH 312: Real Analysis

Name: _____

Date: _____

Directions. Show enough work to support each answer. Cite definitions or established results when you use them in a proof. The six questions cover the major areas on Exam 2.

1. Topology of the Reals

- (a) Define interior point, boundary point, accumulation point, isolated point, open set, and closed set using neighborhoods.
- (b) Let $S = (-2, 0) \cup (0, 2] \cup \{4\}$. Find $\text{int}(S)$, $\text{bd}(S)$, S' , and $\text{cl}(S)$. Is S open? Closed? Identify a boundary point in S , a missing accumulation point, and an isolated point.
- (c) For an arbitrary $F \subseteq \mathbb{R}$, prove from the definitions that F is closed if and only if F^c is open. Explain why $N^*(x; r) \cap F = \emptyset$ and $x \notin F$ imply $N(x; r) \cap F = \emptyset$.

2. Compactness

- (a) Define open cover, finite subcover, and compactness. State Heine–Borel and the sequential characterization of compactness. Classify the following sets, identifying any failed hypothesis:

$$\{0\} \cup [2, 3], \quad [0, 2] \setminus \{1\}, \quad \bigcup_{j \in \mathbb{Z}} [3j, 3j + 1].$$

- (b) Let $K \subseteq \mathbb{R}$ be compact and $F \subseteq K$ be closed in $\mathbb{R}$. Prove from the open-cover definition that F is compact. Explain how to extend an arbitrary open cover of F to cover K and why removing an added set does not uncover any point of F .
- (c) For $A = [0, 2] \setminus \{1\}$ and $n \in \mathbb{N}$, set

$$U_n = (-1, 1 - 1/(n + 1)) \cup (1 + 1/(n + 1), 3).$$

Prove that these open sets cover A but have no finite subcover. Give a sequence in A for which no subsequence has its limit in A . Relate both failures to the missing point.

3. Sequence convergence and uniqueness

- (a) State the definition of $\lim_{n \rightarrow \infty} s_n = L$ and its negation for a fixed L . Prove directly from the definition that

$$\lim_{n \rightarrow \infty} \frac{n + (-1)^n}{2n + 1} = \frac{1}{2}.$$

- (b) Prove that a convergent real sequence has a unique limit. Make your use of the triangle inequality explicit.
- (c) Suppose $\lim_{n \rightarrow \infty} s_n = s$ and $\lim_{n \rightarrow \infty} t_n = t$. Prove from the definition that $\lim_{n \rightarrow \infty} s_n t_n = st$. Use boundedness of a convergent sequence and the identity

$$s_n t_n - st = s_n(t_n - t) + t(s_n - s).$$

Explain how to choose a common index so that both error terms are small.

- (d) Suppose $\lim_{n \rightarrow \infty} s_n = 2$ and $\lim_{n \rightarrow \infty} t_n = -3$. Determine $\lim_{n \rightarrow \infty} (s_n^2 + t_n)/(2t_n - s_n)$ and justify the use of the quotient rule. Then construct two pairs of sequences, each numerator and denominator tending to zero: one whose quotient tends to -2 , and another whose quotient is bounded but has no limit.

4. Monotone convergence and completeness

- (a) State the monotone convergence theorem. Prove the increasing, bounded-above case using $L = \sup\{s_n : n \in \mathbb{N}\}$. Identify separately the roles of completeness, the supremum property, and monotonicity.
- (b) Set $a_n = \sum_{j=1}^n 1/(j(j+2))$. Use

$$\frac{1}{j(j+2)} = \frac{1}{2} \left(\frac{1}{j} - \frac{1}{j+2} \right)$$

- to find an upper bound for (a_n) . Explain why the sequence is increasing, establish convergence, and then find its limit.
- (c) Construct a bounded divergent sequence taking exactly three distinct values, and a decreasing sequence that is unbounded below. Explain which hypothesis fails in each case. Why must convergence be established before using limit laws to solve an equation for a proposed limit?

5. Cauchy sequences and Bolzano–Weierstrass

- (a) Define a Cauchy sequence. Prove that every real Cauchy sequence is bounded, accounting for both the tail and the finite initial segment.
- (b) Suppose (s_n) is Cauchy and $\lim_{k \rightarrow \infty} s_{n_k} = L$ for a subsequence. Prove directly that $\lim_{n \rightarrow \infty} s_n = L$, using one sufficiently late subsequence term and the triangle inequality. State the set and sequence forms of Bolzano–Weierstrass and explain how boundedness supplies the subsequence needed for the reverse implication of the Cauchy criterion.
- (c) Prove the other implication: every convergent real sequence is Cauchy. State the resulting equivalence. Is boundedness alone sufficient for a sequence to be Cauchy? Construct a counterexample whose range is exactly $\{-2, 4\}$ and verify that it fails the Cauchy condition.

6. Function limits and the sequential criterion

- (a) For $f : S \rightarrow \mathbb{R}$ and $c \in S'$, define $\lim_{x \rightarrow c} f(x) = L$ and state the sequential criterion. Explain why the sequences must lie in $S \setminus \{c\}$ and why $c \in S'$ matters. Does the value $f(c)$ affect the limit?
- (b) Prove both implications of the sequential criterion. For the forward implication, explain how ε , δ , and N enter the argument. For the converse, negate the function-limit definition and choose $x_n \in S$ with

$$0 < |x_n - c| < 1/n, \quad |f(x_n) - L| \geq \varepsilon_0.$$

Explain why this contradicts the proposed sequential condition.

- (c) Suppose $f, g : S \rightarrow \mathbb{R}$, $c \in S'$, $\lim_{x \rightarrow c} f(x) = A$, and $\lim_{x \rightarrow c} g(x) = B$. Use the sequential criterion and Problem 3(c) to prove

$$\lim_{x \rightarrow c} f(x)g(x) = AB.$$

Explain why the input sequence must be arbitrary. Using the sequence sum and product laws, apply the same reasoning to find $\lim_{x \rightarrow 1} (x^2 + x)$. Verify this limit directly with an ε - δ proof, first restricting $|x - 1| < 1$.

- (d) Let $h(x) = x/|x|$ for $x \neq 0$. Construct two sequences of nonzero inputs tending to 0 whose image sequences have different limits. Use the sequential criterion to prove that $\lim_{x \rightarrow 0} h(x)$ does not exist. Why is checking only one input sequence insufficient to establish a function limit?