

# Exam 1 Review Worksheet

## MATH 312: Real Analysis

Name: \_\_\_\_\_

Date: \_\_\_\_\_

**Directions.** Show enough work to support each answer. Write all proofs in complete sentences, and cite definitions or established results when you use them. Begin the worksheet in class and complete any unfinished parts as take-home review. The six questions correspond to the six major areas covered on Exam 1.

### 1. Logic and proof techniques.

- (a) Negate the following statement. Move the negation through every quantifier, and do not begin your final answer with “It is not true that.”

For every  $x \in \mathbb{R}$ , there exists  $n \in \mathbb{N}$  such that  $x < n$ .

- (b) Write the converse and the contrapositive of the statement

If  $mn$  is odd, then both  $m$  and  $n$  are odd,  $m, n \in \mathbb{Z}$ .

Which one is logically equivalent to the original statement?

- (c) Prove the original statement. State the proof method you use, and justify each step using the definitions of even and odd integers.

## 2. Sets and functions.

(a) Let  $A, B, C$  be subsets of a universal set  $U$ . Prove by double inclusion that

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

(b) Let  $f : X \rightarrow Y$ , and let  $C, D \subseteq Y$ . Prove that

$$f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D).$$

(c) Let  $E, F \subseteq X$ . Prove that

$$f(E \cap F) \subseteq f(E) \cap f(F).$$

Give an example in which the inclusion is strict, and explain how the failure of equality is connected to the failure of injectivity.

### 3. The natural numbers, recursive arithmetic, and induction.

Recall that  $0 = \emptyset$ ,  $S(n) = n \cup \{n\}$ , and

$$n + 0 = n, \quad n + S(m) = S(n + m),$$

- (a) Write  $1, 2, 3$  explicitly as sets. Recall that  $x < y$  means  $x \in y$ . Prove the transitive property of order:

$$x < y \text{ and } y < z \implies x < z.$$

- (b) Fix  $a, b \in \mathbb{N}$ . Use PMI on  $c$  and the recursive definition of addition to prove

$$(a + b) + c = a + (b + c)$$

for every  $c \in \mathbb{N}$ . Clearly identify  $P(c)$ , the basis, the induction hypothesis, and every use of the recursive definition.

- (c) Use PMI to prove the standard discrete sum formula

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

for every  $n \in \mathbb{N}$ . Clearly identify the statement, basis, induction hypothesis, and induction step.

4. **Equivalence classes and number-system extensions.**

- (a) The integers are equivalence classes of pairs in  $\mathbb{N} \times \mathbb{N}$ , where

$$(a, b) \sim (c, d) \iff a + d = b + c.$$

Prove that this relation is reflexive, symmetric, and transitive. Explain why this is necessary before defining

$$\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/\sim.$$

- (b) The rational numbers are equivalence classes of pairs in  $\mathbb{Z} \times \mathbb{N}$ , where

$$(a, b) \sim (c, d) \iff ad = bc.$$

Prove that this relation is reflexive, symmetric, and transitive. In the transitivity argument, identify where you use the fact that the denominators belong to  $\mathbb{N}$  and are therefore nonzero.

- (c) Explain why equivalence classes, rather than individual ordered pairs, are used in both constructions. Then state the principal algebraic feature gained in passing from  $\mathbb{N}$  to  $\mathbb{Z}$ , and the feature gained in passing from  $\mathbb{Z}$  to  $\mathbb{Q}$ .

## 5. Ordered-field properties.

Recall that rational numbers are equivalence classes

$$[(a, b)] = \{(c, d) \in \mathbb{Z} \times \mathbb{N} : ad = bc\},$$

where  $a \in \mathbb{Z}$  and  $b \in \mathbb{N}$ , and that

$$[(a, b)] + [(c, d)] = [(ad + bc, bd)].$$

- (a) Show how commutativity of addition is inherited through the constructions. First, for  $x = [(a, b)]$  and  $y = [(c, d)]$  in  $\mathbb{Z}$ , prove  $x + y = y + x$  using commutativity of addition in  $\mathbb{N}$ . Then, for  $p = [(r, s)]$  and  $q = [(t, u)]$  in  $\mathbb{Q}$ , prove  $p + q = q + p$  using the arithmetic properties of  $\mathbb{Z}$ .
- (b) Let

$$x = [(a, b)], \quad y = [(c, d)], \quad z = [(e, f)], \quad b, d, f \in \mathbb{N}.$$

Recall that  $x < y$  means  $ad < bc$ . Suppose  $x < y$ . Use the definitions of rational addition and order to prove

$$x + z < y + z.$$

- (c) State all of the field properties and all of the order properties that together make  $\mathbb{Q}$  an ordered field. Explain why requiring positive denominators is essential when the order is defined by cross multiplication.

## 6. Cauchy sequences, the real numbers, and completeness.

- (a) State precisely what it means for a sequence  $(x_n)$  in  $\mathbb{Q}$  to be Cauchy with respect to  $d(x, y) = |x - y|$ . For rational Cauchy sequences, define

$$(x_n) \sim (y_n) \iff (x_n - y_n) \text{ is a null sequence.}$$

Prove that  $\sim$  is an equivalence relation. In the transitivity proof, make the  $\varepsilon/2$  argument explicit.

- (b) Let  $S \subseteq \mathbb{R}$  be nonempty and bounded above, and let  $s = \sup S$ . Prove that, for every  $\varepsilon > 0$ , there exists  $x \in S$  such that

$$s - \varepsilon < x \leq s.$$

Clearly indicate where you use the definition of supremum.

- (c) Define the rational partial sums

$$s_n = \sum_{k=0}^n \frac{(-1)^k}{2k+1}.$$

Compute  $s_0, s_1, s_2$ , and explain why every  $s_n$  is rational. You may use the estimate

$$\left| s_n - \frac{\pi}{4} \right| \leq \frac{1}{2n+3}$$

and the fact that  $\pi$  is irrational. Use the estimate to prove that  $(s_n)$  is Cauchy. Explain why  $(s_n)$  does not converge in  $\mathbb{Q}$ , although it converges in  $\mathbb{R}$ , and state what this reveals about completeness.