Real Numbers

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1 Sequences of Rational Numbers

A *rational sequence* is a function $s\colon \mathbb{N} \to \mathbb{Q}$ We denote the nth element of the sequence by $s(n) = s_n$ We may also denote the sequence by

$$(s_n)_{n=1}^{\infty} = (s_1, s_2, s_3, \ldots).$$

A *rational sequence of partial sums* is a sequence where the nth term is defined by the sum of the first n terms in a rational sequence, that is,

$$s_n = \sum_{k=1}^{n} a_k,$$

where $a_k \in \mathbb{Q}$ for all $k \in \mathbb{N}$.

A rational sequence $s\colon \mathbb{N} \to \mathbb{Q}$ is a *Cauchy sequence* if for all rational $\epsilon > 0$ there is a rational N such that

$$m, n > N \Rightarrow |s_n - s_m| < \epsilon.$$

As an example, $(1/n)_{n=1}^{\infty}$ is a sequence of rational numbers. This rational sequence is a Cauchy sequence since for all rational $\epsilon > 0$, $N = 2/\epsilon$ satisfies

$$m, n > N \Rightarrow \left|\frac{1}{n} - \frac{1}{m}\right| < \epsilon.$$

Moreover, this sequence coverges to 0. As another example, consider the following rational sequence of partial sums

$$\left(\sum_{k=1}^{n} \frac{(-4)^{k+1}}{2k-1}\right)_{n=1}^{\infty}.$$

By the alternating series test, this sequence of partial sums converges over the reals and is therefore a Cauchy sequence. However, this sequence converges to π, which is an example of an *irrational number*, that is, a real number that cannot be represented as a fraction of integers. The irrational numbers are what's missing from the rational numbers making them insufficient for the study of calculus.

Lemma 1.1 shows that every rational Cauchy sequence is bounded, which is used to esablish the sum and product of rational Cauchy sequences in Theorem 1.2.

Lemma 1.1. *Every rational Cauchy sequence is bounded.*

Proof. Let $s\colon \mathbb{N} \to \mathbb{Q}$ be a rational Cauchy sequence. Let $\epsilon = 1$. Then, there exists a $N \in \mathbb{N}$ such that

$$m, n > N \Rightarrow |s_n - s_m| < 1.$$

. Now, fix $m > N$. Then, $|s_n| < |s_m| + 1$ for all $n > N$. Define

$$M = \max\{|s_1|, |s_2|, \ldots, |s_N|, |s_m| + 1\}.$$

Then, $|s_n| \le M$ for all $n \in \mathbb{N}$. Therefore, the sequence s is bounded. □

Theorem 1.2. *Let $x\colon \mathbb{N} \to \mathbb{R}$ and $y\colon \mathbb{N} \to \mathbb{R}$. Then, $x + y = (x_n + y_n)_{n=1}^{\infty}$ and $x \cdot y = (x_n \cdot y_n)_{n=1}^{\infty}$ are Cauchy sequences.*

2 Properties of the Real Numbers

The *real numbers* are defined by

$$\mathbb{R} = \{\pm d_0.d_1d_2d_3\ldots : 0 \leq d_i \leq 9\}.$$

Given the set of rational Cauchy sequences, the set of real numbers can be constructed using equivalence classes. Let S denote the set of all rational Cauchy sequences. Define the equivalence relation R on the set $S \times S$ as follows

$$((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}) \in R \Leftrightarrow \lim_{n\to\infty}(x_n - y_n) = 0,$$

where $\lim_{n\to\infty}(x_n - y_n) = 0$ if for all $\epsilon > 0$ there exists an N such that $n > N \Rightarrow |x_n - y_n| < \epsilon$. Then, each real number corresponds to an equivalence class with respect to R. For example, the number 0 corresponds to the equivalence class

$$E_{(1/n)_{n=1}^{\infty}} = \left\{ s \in S \colon \lim_{n\to\infty}(s_n - 1/n) = 0 \right\}.$$

Similarly, the number 1 corresponds to the equivalence class $E_{(1-1/n)_{n=1}^{\infty}}$ and the number -1 corresponds to the equivalence class $E_{(-1-1/n)_{n=1}^{\infty}}$. In general, given any number $d = d_0.d_1d_2d_3\cdots$, the sequence

$$x = (d_0.d_1, d_0.d_1d_2, d_0.d_1d_2d_3, \ldots)$$

generates the equivalence class E_x corresponding to the number d.

In what follows, we use the sum and product of rational Cauchy sequences described in Theorem 1.2 to define the sum and product of real numbers. Let d and d' denote real numbers corresponding to the equivalence classes E_x and $E_{x'}$, respectively. Then, we define the addition operation $d + d'$ by the real number corresponding to the equivalence class $E_{x+x'}$, where $x + x'$ denotes the sum of rational Cauchy sequences. Moreover, we define the multiplication operation $d \cdot d'$ by the real number corresponding to the equivalence class $E_{x\cdot x'}$, where $x \cdot x'$ denotes the product of rational Cauchy sequences.

Note that the subtraction operation $d - d'$ is defined by $d + (-d')$, where $-d'$ is the real number corresponding to the equivalence class $E_{-x'}$ and $-x' = (-x'_n)_{n=1}^{\infty}$ is a rational Cauchy sequence. Moreover, if $d' \neq 0$, then the sequence x' can be selected to have only non-zero terms. Hence, we define the division operation d/d', where $d' \neq 0$, by $d \cdot (1/d')$, where $1/d'$ is the real number corresponding to the equivalence class $E_{1/x'}$ and $1/x' = (1/x'_n)_{n=1}^{\infty}$ is a rational Cauchy sequence.

From these definitions it can be shown that the real numbers satisfy the following properties.

(a) Closure: Given any $a, b \in \mathbb{R}$, $a + b \in \mathbb{R}$ and $a \cdot b \in \mathbb{R}$.

(b) Commutative: Given any $a, b, \in \mathbb{R}$, $a + b = b + a$ and $a \cdot b = b \cdot a$.

(c) Associative: Given any $a, b, c \in \mathbb{R}$, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(d) Distributive: Given any $a, b, c \in \mathbb{R}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

(e) Identity: There exists $0, 1 \in \mathbb{R}$ such that $a + 0 = a$ and $a \cdot 1 = a$ for all $a \in \mathbb{R}$.

(f) Additive Inverse: For every $a \in \mathbb{R}$, there exists $-a \in \mathbb{R}$ such that $a + (-a) = 0$.

(g) Multiplicative Inverse: For every $a \in \mathbb{R} \setminus \{0\}$, there exists a $1/a \in \mathbb{R}$ such that $a \cdot (1/a) = 1$.

Therefore, the real numbers are a field.

3 Ordering

We can use the ordering of the rational numbers to define an ordering of the real numbers. Let $x, y \in S$ denote rational Cauchy sequences. Then, we write $x < y$ if there exists a rational $\epsilon > 0$ and a rational N such that

$$n > N \Rightarrow x_n < y_n - \epsilon.$$

Also, we write $x = y$ if for all rational $\epsilon > 0$ there is a rational N such that

$$n > N \Rightarrow |x_n - y_n| < \epsilon.$$

Finally, we write $x > y$ if there exists a rational $\epsilon > 0$ and a rational N such that

$$n > N \Rightarrow x_n > y_n + \epsilon.$$

This ordering of rational Cauchy sequences implies an ordering of real numbers. Therefore, the real numbers form an ordered field.

4 Metric

We can define a metric on the set of real numbers just as we did with the set of rational numbers. First, we define the absolute value of $x \in \mathbb{R}$ by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Then, we define the distance between $x, y \in \mathbb{R}$ by $d(x, y) = |x - y|$. As with the rational numbers, this distance function satisfies the non-negativity, identity, symmetry, and triangle inequality properties. Therefore, the real numbers are a metric space.