Rational Numbers

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1 Properties of the Integers

The *integers* are defined by

$$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}.$$

Given the natural numbers $\mathbb{N}$, the set of integers can be constructed using equivalence classes. In particular, define the equivalence relation R on the set $\mathbb{N} \times \mathbb{N}$ as follows

$$((a, b), (c, d)) \in R \Leftrightarrow a + d = b + c.$$

Then, each integer corresponds to an equivalence class with respect to R. For example, the integer 0 corresponds to the equivalence class

$$E_{(1,1)} = \{(a, b) \colon a + 1 = b + 1\}.$$

Similarly, the integer 1 corresponds to the equivalence class $E_{(2,1)}$, and the integer -1 corresponds to the equivalence class $E_{(1,2)}$. In general, given a natural number $n \in \mathbb{N}$, the equivalence class $E_{(n+1,1)}$ corresponds to the integer n and the equivalence class $E_{(1,n+1)}$ corresponds to the integer $-n$.

Let m and n denote integers corresponding to the equivalence classes $E_{(a,b)}$ and $E_{(c,d)}$, respectively. Then, we define the addition operation $m+n$ by the integer corresponding to the equivalence class $E_{(a+c,b+d)}$. Note that the subtraction operation $m - n$ is defined by $m + (-n)$, where $-n$ is the integer corresponding to the equivalence class $E_{(d,c)}$. Moreover, we define the multiplication operation $m \cdot n$ by the integer corresponding to the equivalence class $E_{(a\cdot c+b\cdot d,a\cdot d+b\cdot c)}$.

From these definitions it can be shown that the integers satisfy the following properties.

(a) Closure: Given any $a, b \in \mathbb{Z}$, $a + b \in \mathbb{Z}$ and $a \cdot b \in \mathbb{Z}$.

(b) Commutative: Given any $a, b, \in \mathbb{Z}$, $a + b = b + a$ and $a \cdot b = b \cdot a$.

(c) Associative: Given any $a, b, c \in \mathbb{Z}$, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(d) Distributive: Given any $a, b, c \in \mathbb{Z}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

(e) Identity: There exists $0, 1 \in \mathbb{Z}$ such that $a + 0 = a$ and $a \cdot 1 = a$ for all $a \in \mathbb{Z}$.

2 Properties of the Rational Numbers

The *rational numbers* are defined by

$$\mathbb{Q} = \left\{ \frac{p}{q} \colon p \in \mathbb{Z},\ q \in \mathbb{N} \right\}.$$

Given the integers $\mathbb{Z}$, the set of rational numbers can be constructed using equivalence classes. In particular, define the equivalence relation R on the set $\mathbb{Z} \times \mathbb{N}$ as follows

$$((a, b), (c, d)) \in R \Leftrightarrow a \cdot d = b \cdot c.$$

Then, each rational number corresponds to an equivalence class with respect to R. For example, the rational number $2/3$ corresponds to the equivalence class

$$E_{(2,3)} = \{(a,b) \colon 3 \cdot a = 2 \cdot b\}.$$

Similarly, the ratoinal number 0 corresponds to the equivalence class $E_{(0,1)}$, and the rational number 1 corresponds to the equivalence class $E_{(1,1)}$. In general, given integers $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, the equivalence class $E_{(p,q)}$ corresponds to the rational number p/q.

Let p/q and p'/q' denote rational numbers corresponding to the equivalence classes $E_{(p,q)}$ and $E_{(p',q')}$, respectively. Then, we define the addition operation $p/q + p'/q'$ by the rational number corresponding to the equivalence class $E_{(p\cdot q'+p'\cdot q, q\cdot q')}$. Note that the subtraction operation $p/q - p'/q'$ is defined by $p/q + (-p'/q')$, where $-p'/q'$ is the rational number corresponding to the equivalence class $E_{(-p',q')}$. Moreover, we define the multiplication opertation $(p/q)\cdot(p'/q')$ by the rational number corresponding to the equivalence class $E_{(p\cdot p', q\cdot q')}$. Note that the division operation $(p/q)/(p'/q')$, where $p' \neq 0$, is defined by $(p/q)\cdot(q'/p')$, where q'/p' is the rational number corresponding to the equivalence class $E_{(q',p')}$.

From these definitions it can be shown that the rational numbers satisfy the following properties.

(a) Closure: Given any $a, b \in \mathbb{Q}$, $a + b \in \mathbb{Q}$ and $a \cdot b \in \mathbb{Q}$.

(b) Commutative: Given any $a, b, \in \mathbb{Q}$, $a + b = b + a$ and $a \cdot b = b \cdot a$.

(c) Associative: Given any $a, b, c \in \mathbb{Q}$, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(d) Distributive: Given any $a, b, c \in \mathbb{Q}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

(e) Identity: There exists $0, 1 \in \mathbb{Q}$ such that $a + 0 = a$ and $a \cdot 1 = a$ for all $a \in \mathbb{Q}$.

(f) Additive Inverse: For every $a \in \mathbb{Q}$, there exists $-a \in \mathbb{Q}$ such that $a + (-a) = 0$.

(g) Multiplicative Inverse: For every $a \in \mathbb{Q} \setminus \{0\}$, there exists a $1/a \in \mathbb{Q}$ such that $a \cdot (1/a) = 1$.

It is important to note that sets that satisfy the above properties are called *fields*.

3 Ordering

In addition to the field properties, the rational numbers have an *ordering*. That is, there exists a relation $\leq$ on $\mathbb{Q}$ that satisfies the following properties.

(a) Trichotomy: For all $x, y \in \mathbb{Q}$, exactly one of the relations hold $x = y$, $x > y$, or $x < y$.

(b) Transitivity: For all $x, y, z \in \mathbb{Q}$, if $x < y$ and $y < z$ then $x < z$.

(c) Additivity: For all $x, y, z \in \mathbb{Q}$, if $x < y$ then $x + z < y + z$.

(d) Multiplicativity: For all $x, y, z \in \mathbb{Q}$, if $x < y$ and $z > 0$ then $xz < yz$.

Since the rational numbers are a field with an ordering we say that $\mathbb{Q}$ is an *ordered field*.

This ordering is first defined for the natural numbers. Recall that $\mathbb{N}$ is the smallest set that contains $1 = \{\emptyset\}$ and is closed under the successor function. Given $a, b \in \mathbb{N}$, we write $a < b$ if $a \subsetneq b$, $a = b$ if $a \subseteq b$ and $b \subseteq a$, and $a > b$ if $b \subsetneq a$.

Once the ordering on the natural numbers is defined, we can use it to define an ordering on the integers. Recall that $\mathbb{Z}$ corresponds to the equivalence classes of $R \subseteq \mathbb{N} \times \mathbb{N}$, where

$$((a,b),(c,d)) \in R \Leftrightarrow a + d = b + c.$$

Let $m, n \in \mathbb{Z}$ corresponding to the equivalence classes $E_{(a,b)}$ and $E_{(c,d)}$, respectively. Then, we write $m < n$ if $a + d < b + c$, $m = n$ if $a + d = b + c$, and $m > n$ if $a + d > b + c$.

Once the ordering on the integers is defined, we can use it to define an ordering on the rational numbers. Recall that $\mathbb{Q}$ corresponds to the equivalence classes of $R \subseteq \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$, where

$$((a, b), (c, d)) \in R \Leftrightarrow a \cdot d = b \cdot c.$$

Let p/q and p'/q' denote rational numbers corresponding to the equivalence classes $E_{(p,q)}$ and $E_{(p',q')}$, respectively. Then, we write $p/q < p'/q'$ if $p \cdot q' < p' \cdot q$, $p/q = p'/q'$ if $p \cdot q' = p' \cdot q$, and $p/q > p'/q'$ if $p \cdot q' > p' \cdot q$.

4 Metric

This ordering allows us to define a *metric*, that is, a notion of distance between rational numbers. First, we define the absolute value of $x \in \mathbb{Q}$ by

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Then, we define the distance between $x, y \in \mathbb{Q}$ by $d(x, y) = |x - y|$. Note that this distance function has several important properties.

(a) Non-negativity: $d(x, y) \geq 0$ for all $x, y \in \mathbb{Q}$.

(b) Identity: $d(x, y) = 0$ if and only if $x = y$.

(c) Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in \mathbb{Q}$.

(d) Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathbb{Q}$.

Since the rational numbers have this metric, $\mathbb{Q}$ is known as a *metric space*.