Proof Techniques

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1 Logical Statements

The language of mathematics consists primarily of declarative sentences. If a sentence can be classified as true or false, it is called a *statement*. Let p and q denote statements. There are several connectives to combine the statements p and q to form new statements. The *negation* of p, denoted $\neg p$, represents the logical opposite of p. The *conjunction* of p and q, denoted $p \wedge q$, represents the statement p *and* q. The *disjunction* of p and q, denoted $p \vee q$, represents the statement p *or* q. A statement of the form if p then q is called an *implication* or *conditional* statement. It asserts that if the *hypothesis* p is true, then the *conclusion* q must also be true. We denote the implication by $p \Rightarrow q$. We summarize the truth values of these connective statements in Table 1.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \Rightarrow q$
T	T	F	T	T	T
T	F	F	F	T	F
F	T	T	F	T	T
F	F	T	F	F	T

Table 1: Truth table for logical connectives

When a statement involves a variable, we use quantifiers to clarify the domain of the variable. In particular, the *existential quantifier* $\exists$ indicates that there is at least one value for the variable such that the statement holds. The *universal quantifier* $\forall$ indicates that the statement holds for all values of the variable within the given domain. For example,

$$\exists x \in \mathbb{R} \ \ni \ x^2 - 5x + 6 = 0$$

is read "there exists a real x such that $x^2 - 5x + 6 = 0$." This statement is true since the values $x = 2$ and $x = 3$ satisfy. In contrast, the statement

$$\forall x \in \mathbb{R}, \ x^2 - 5x + 6 = 0$$

is read "for all real x, $x^2 - 5x + 6 = 0$." This statement is false since it does not hold for the value $x = 1$, such an example is called a *counter example*.

Quantified statements can be confusing when there are two or more quantifiers in the same statement. For example, consider the statement "for every $\epsilon > 0$, there exists a $\delta > 0$ such that $1-\delta < x < 1+\delta$ implies that $5 - \epsilon < 2x + 3 < 5 + \epsilon$." This statement can be written as

$$\forall \epsilon > 0, \ \exists \delta > 0 \ \ni \ (1 - \delta < x < 1 + \delta) \Rightarrow (5 - \epsilon < 2x + 3 < 5 + \epsilon) \, .$$

Note that the order of the quantifiers matters; in particular, this statement claims that for every $\epsilon > 0$ we can find a $\delta > 0$ such that an implication statement holds.

2 Proof Techniques

When proving an implication statement, $p \Rightarrow q$, there are several techniques we may use. A *direct proof* starts by assuming the hypothesis p is true and uses logical reasoning to deduce q. We can also prove the implication by proving logically equivalent statements directly. For example, the *contrapositive*, denoted $\neg q \Rightarrow \neg p$, is logically equivalent to $p \Rightarrow q$. Similarly, if c denotes a statement that is always false, then the *contradiction*, denoted $(p \wedge \neg q) \Rightarrow c$, is logically equivalent to $p \Rightarrow q$. We summarize the truth values of these statements in Table 2.

p	q	c	$p \Rightarrow q$	$\neg q \Rightarrow \neg p$	$p \wedge \neg q$	$(p \wedge \neg q) \Rightarrow c$
T	T	F	T	T	F	T
T	F	F	F	F	T	F
F	T	F	T	T	F	T
F	F	F	T	T	F	T

Table 2: Truth table for statements logically equivalent to the implication statement.

In the following propositions, we illustrate these proof techniques. We begin with a direct proof.

Proposition 2.1. *For every $\epsilon > 0$, there exists a $\delta > 0$ such that $1 - \delta < x < 1 + \delta$ implies that $5 - \epsilon < 2x + 3 < 5 + \epsilon$.*

Proof. Let $\epsilon > 0$ and let $\delta = \epsilon/2$. Then, $\delta > 0$ and the following implications hold

$$\begin{aligned} 1 - \delta < x < 1 + \delta &\Rightarrow 2 - 2\delta < 2x < 2 + 2\delta \\ &\Rightarrow 5 - 2\delta < 2x + 3 < 5 + 2\delta \\ &\Rightarrow 5 - \epsilon < 2x + 3 < 5 + \epsilon. \end{aligned}$$

□

Next, we use a proof by contrapositive.

Proposition 2.2. *Let $f(x)$ be a continuous function on $[0, 1]$. If $\int_0^1 f(x)dx \neq 0$, then there exists an x in $[0, 1]$ such that $f(x) \neq 0$.*

Proof. We proceed via a proof by contrapositive. Suppose that for all x in $[0, 1]$, $f(x) = 0$. Then, the region bounded by the curves $y = f(x)$, $y = 0$, $x = 0$, and $x = 1$ has zero area. Therefore, $\int_0^1 f(x)dx = 0$. □

Finally, we use a proof by contradiction.

Proposition 2.3. *Let x be a real number. If $x > 0$, then $1/x > 0$.*

Proof. We proceed via a proof by contradiction. Suppose that $x > 0$ and $1/x \leq 0$. Then,

$$x\frac{1}{x} \leq x(0),$$

which implies the contradiction $1 \leq 0$. □