Natural Numbers

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1 Properties of the Natural Numbers

The *natural numbers* are defined by

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

In set theory, the natural numbers are constructed recursively. Starting with $0 = \emptyset$, the successor of n is defined by $S(n) = n \cup \{n\}$. For example,

$$\begin{aligned}
1 &= S(0) = \emptyset \cup \{\emptyset\} = \{\emptyset\}, \\
2 &= S(1) = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}, \\
3 &= S(2) = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}.
\end{aligned}$$

The set of natural numbers $\mathbb{N}$ is the smallest set containing 1 that is closed under the successor function.

Note that addition and multiplication of natural numbers can be defined recursively using the successor function. The addition of natural numbers is defined by

$$n + 0 = n, \ n + S(m) = S(n + m) \ (\forall m \geq 0)$$

and the multiplication of natural numbers is defined by

$$n \cdot 0 = 0, \ n \cdot S(m) = n \cdot m + n \ (\forall m \geq 0).$$

For example,

$$\begin{aligned}
2 + 3 &= 2 + S(2) = S(2 + 2) \\
&= S(2 + S(1)) = S(S(2 + 1)) \\
&= S(S(2 + S(0))) = S(S(S(2 + 0))) \\
&= S(S(S(2))) = 5,
\end{aligned}$$

and

$$\begin{aligned}
2 \cdot 3 &= 2 \cdot S(2) = 2 \cdot 2 + 2 \\
&= 2 \cdot S(1) + 2 = 2 \cdot 1 + 2 + 2 \\
&= 2 \cdot S(0) + 2 + 2 = 2 \cdot 0 + 2 + 2 + 2 \\
&= 0 + 2 + 2 + 2 = 6,
\end{aligned}$$

where the last line is computed via the recursive definition for addition.

From these definitions it can be shown that the natural numbers satisfy the following properties.

(a) Closure: Given any $a, b \in \mathbb{N}$, $a + b \in \mathbb{N}$ and $a \cdot b \in \mathbb{N}$.

(b) Commutative: Given any $a, b, \in \mathbb{N}$, $a + b = b + a$ and $a \cdot b = b \cdot a$.

(c) Associative: Given any $a, b, c \in \mathbb{N}$, $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(d) Distributive: Given any $a, b, c \in \mathbb{N}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

(e) Well-Ordering: Given any non-empty $S \subseteq \mathbb{N}$, there is a $m \in S$ such that $m \leq k$ for all $k \in S$.

2 Principle of Mathematical Induction

The definition of the natural numbers naturally lends itself to the principle of mathematical induction, which allows us to prove that certain statements are true for all natural numbers.

Theorem 2.1 (Principle of Mathematical Induction)**.** *Let $P(n)$ be a statement that is either true or false for all $n \in \mathbb{N}$. Suppose that*

(a) $P(1)$ is true,

(b) for each $k \in \mathbb{N}$, if $P(k)$ is true then $P(k+1)$ is true.

Then, $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Define

$$S = \{n \in \mathbb{N} \colon P(n) \text{ is true}\}.$$

Then, by (a), $1 \in S$. Moreover, by (b), S is closed under the successor function. Since the natural numbers is the smallest set that contains 1 and is closed under the successor function, it follows that $\mathbb{N} \subseteq S$. Since $S \subseteq \mathbb{N}$, by definition, it follows that $S = \mathbb{N}$. Therefore, $P(n)$ is true for all $n \in \mathbb{N}$. □

Regarding Theorem 2.1, it is common to refer to the verification of part (a) as the *basis for induction* and part (b) as the *induction step.* The assumption that $P(k)$ is true in part (b) is known as the *induction hypothesis.* As an example of the application of the principle of mathematical induction, consider the following proposition.

Proposition 2.2. *For every $n \in \mathbb{N}$,*

$$1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1).$$

Proof. For the basis of induction, note that $1 = \frac{1(2)}{2}$. For the induction step, let $k \in \mathbb{N}$ and suppose that

$$1 + 2 + 3 + \cdots + k = \frac{1}{2}k(k+1).$$

Adding $(k+1)$ to both sides gives

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= \frac{1}{2}k(k+1) + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2}. \end{aligned}$$

Therefore, if the statement holds for $k \in \mathbb{N}$ then the statement holds for the successor $(k+1)$. By the prinicple of mathematical induction, the statement holds for all $n \in \mathbb{N}$. □

We can use the principle of mathematical induction to prove the well-ordering property of the natural numbers.

Theorem 2.3. *Let $S \subset \mathbb{N}$ be non-empty. Then, there is a $m \in S$ such that $m \leq k$ for all $k \in S$.*

Proof. We proceed via induction on the cardinality of S. For the basis of induction, suppose that $|S| = 1$. Then, there is exactly one element in S, which we denote by m. Moreover, it is clear that $m \leq k$ for all $k \in S$. For the induction step, let $k \in \mathbb{N}$ and suppose that the well-ordering property holds for all subsets of cardinality k. Now, let S be a subset of cardinality $(k+1)$. Then, S can be partitioned as $S = S' \cup \{a\}$, for some $a \in \mathbb{N}$. By the induction hypothesis, there exists a $m \in S'$ such that $m \leq k$ for all $k \in S'$. Set $m' = \min\{m, a\}$. Then, $m' \leq k$ for all $k \in S$. Therefore, by Theorem 2.1 the well-ordering property for all non-empty subsets of natural numbers. □