The Mean Value Theorem

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1 The Derivative

We define the derivative of a function at the interior points of its domain. Since the interior points of a set are those that have a neighborhood contained in the set, we will focus on functions defined over intervals. In particular, we define the derivative of $f\colon [a,b] \to \mathbb{R}$ at a point $c \in (a,b)$ by

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c},$$

provided the limit exists and is finite. In this case, we say that f is differentiable at c.

Equivalentyly, we say that f is differentiable at c if and only if for all $x\colon \mathbb{N} \to [a,b] \setminus \{c\}$ such that $\lim_{n\to\infty} x_n = c$, the sequence

$$\frac{f(x_n) - f(c)}{x_n - c}$$

converges. Moreover, we call the limiting value of this sequence the derivative of f at c, which we denote by $f'(c)$.

We say that f is differentiable on (a,b) if f is differentiable at every $c \in (a,b)$.

2 Rolle's Theorem

We begin with the definition of a relative extrema. Let $S \subseteq \mathbb{R}$ and $f\colon S \to \mathbb{R}$. Then, $c \in S$ is a relative max if there exists a $\delta \in \mathbb{R}_{>0}$ such that

$$x \in N(c;\delta) \cap S \Rightarrow f(x) \leq f(c).$$

Similarly, $c \in S$ is a relative min if there exists a $\delta \in \mathbb{R}_{>0}$ such that

$$x \in N(c;\delta) \cap S \Rightarrow f(x) \geq f(c).$$

If c is either a relative max or min, then we say that c is a relative extrema of f. The following result shows that differentiable functions have a zero derivative at relative extrema.

Lemma 2.1. *Let $f\colon [a,b] \to \mathbb{R}$ be differentiable on (a,b). If $c \in (a,b)$ is a relative extrema of f, then $f'(c) = 0$.*

Proof. Without loss of generality, assume that $c \in (a,b)$ is a relative max of f. Fix $\delta \in \mathbb{R}_{>0}$ such that

$$x \in N(c;\delta) \cap (a,b) \Rightarrow f(x) \leq f(c).$$

Let $x\colon \mathbb{N} \to N(c;\delta) \cap (a,b) \setminus \{c\}$ be an increasing sequence that converges to c. Then,

$$\frac{f(x_n) - f(c)}{x_n - c} \geq 0,$$

for all $n \in \mathbb{N}$. Also, let $y\colon \mathbb{N} \to N(c;\delta) \cap (a,b) \setminus \{c\}$ be a decreasing sequence that converges to c. Then,

$$\frac{f(y_n) - f(c)}{y_n - c} \leq 0,$$

for all $n \in \mathbb{N}$. Since both sequences converges to $f'(c)$, it follows that $f'(c) = 0$. $\square$

The following result is known as Rolle's theorem.

Theorem 2.2. *Let* $f\colon [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b)*. Suppose that* $f(a) = f(b)$*. Then, there is a* $c \in (a,b)$ *such that* $f'(c) = 0$*.*

Proof. Since $[a,b]$ is compact and f is continuous, it follows that $f([a,b])$ is compact. So, there exists $x_1, x_2 \in [a,b]$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in [a,b]$. If x_1 and x_2 are both endpoints, then

$$f(x) = f(a) = f(b)$$

for all $x \in [a,b]$. Therefore, f is constant on $[a,b]$, so $f'(x) = 0$ for all $x \in (a,b)$. If either x_1 or x_2 are not endpoints, then they are relative extrema and it follows that $f'(x_1) = 0$ or $f'(x_2) = 0$. □

3 Mean Value Theorem

We are now ready to prove the mean value theorem, which is one of the most important results in differential calculus.

Theorem 3.1. *Let* $f\colon [a,b] \to \mathbb{R}$ *be continuous on* $[a,b]$ *and differentiable on* (a,b)*. Then, there exists a* $c \in (a,b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Define $g\colon [a,b] \to \mathbb{R}$ by

$$g(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Then, g is continuous on $[a,b]$ and differentiable on (a,b). Moreover, $h\colon [a,b] \to \mathbb{R}$ defined by $h(x) = f(x) - g(x)$ is continuous on $[a,b]$ and differentiable on (a,b). Since $h(a) = h(b) = 0$, Rolle's theorem implies that there exists a $c \in (a,b)$ such that $h'(c) = 0$. Furthermore, $h'(c) = f'(c) - g'(c)$ and it follows that

$$f'(c) = g'(c) = \frac{f(b) - f(a)}{b - a}.$$

□