Limits of Functions

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1 Limits of Functions

Let $f\colon S \to \mathbb{R}$ and let $c \in S'$. We say that the function f has a *limit* at c if there exists a $L \in \mathbb{R}$ where for all $\epsilon \in \mathbb{R}_{>0}$ there is a $\delta \in \mathbb{R}_{>0}$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - c| < \delta$ and $x \in S$. In this definition of the limit, it is important that c is an accumulation point of S since it guarantees that $N^*(c;\delta) \cap S \neq \emptyset$ for all $\delta > 0$. Also, since $0 < |x - c| < \delta$, the limit definition does not consider the value of f at c; in fact, f need not be defined at c.

If f has a limit at c, we call $L \in \mathbb{R}$ the limiting value of f at c. In this case, we write

$$\lim_{x \to c} f(x) = L.$$

We may also say that f converges to L as x approaches c. The following result shows that we can characterize the limit using the language of neighborhoods.

Theorem 1.1. *Let $f\colon S \to \mathbb{R}$ and let $c \in S'$. Then, $\lim_{x\to c} f(x) = L$ if and only if for each neighborhood V of L there exists a deleted neighborhood U of c such that $f(U \cap S) \subseteq V$.*

The following result shows that the limiting value of a function is unique.

Proposition 1.2. *Let $f\colon S \to \mathbb{R}$ and let $c \in S'$. If f has a limit at c, then the limiting value is unique.*

Proof. Suppose that f has a limit at c. Also, suppose that $L_1, L_2 \in \mathbb{R}$ are limiting values of f at c.

Now, let $\epsilon \in \mathbb{R}_{>0}$. Then, there exists $\delta_1, \delta_2 \in \mathbb{R}_{>0}$ such that

$$x \in S \wedge 0 < |x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{\epsilon}{2}$$

and

$$x \in S \wedge 0 < |x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \frac{\epsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, $x \in S \wedge 0 < |x - c| < \delta$ implies that

$$\begin{aligned} |L_1 - L_2| &= |L_1 - f(x) + f(x) - L_2| \\ &\leq |L_1 - f(x)| + |f(x) - L_2| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Since the above inequality holds for any $\epsilon \in \mathbb{R}_{>0}$, it follows that $L_1 = L_2$. □

For example, consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 1$ for all $x \in \mathbb{R}$. We claim that $\lim_{x\to 3} f(x) = 7$. To this end, let $\epsilon \in \mathbb{R}_{>0}$ and define $\delta = \epsilon/2$. Then, $x \in \mathbb{R} \wedge 0 < |x - 3| < \delta$ implies that

$$\begin{aligned} |(2x + 1) - 7| &= |2x - 6| \\ &= 2\,|x - 3| \\ &< 2\delta = \epsilon. \end{aligned}$$

As another example, consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 2x + 6$ for all $x \in \mathbb{R}$. We claim that $\lim_{x\to 3} f(x) = 21$. To this end, let $\epsilon \in \mathbb{R}_{>0}$ and define $\delta = \min\{1, \epsilon/9\}$. Then, $x \in \mathbb{R} \wedge 0 < |x-3| < \delta$ implies that

$$\begin{aligned} \left|(x^2 + 2x + 6) - 21\right| &= \left|x^2 + 2x - 15\right| \\ &= |(x+5)(x-3)| \\ &= |x+5|\,|x-3| \\ &< 9\delta = \epsilon. \end{aligned}$$

2 Sequential Limits

The following theorem establishes an important relationship between limits of functions and limits of sequences.

Theorem 2.1. *Let $f\colon S \to \mathbb{R}$ and let $c \in S'$. Also, let $L \in \mathbb{R}$. Then, $\lim_{x\to c} f(x) = L$ if and only if for every sequence $s\colon \mathbb{N} \to \mathbb{R}$ such that $\operatorname{rng}(s) \subseteq S \setminus \{c\}$ and $\lim_{n\to\infty} s_n = c$, we have*

$$\lim_{n\to\infty} f(s_n) = L.$$

Proof. Suppose that $\lim_{x\to c} f(x) = L$. Let $s\colon \mathbb{N} \to \mathbb{R}$ be a sequence such that $\operatorname{rng}(s) \subseteq S \setminus \{c\}$ and $\lim_{n\to\infty} s_n = c$. Let $\epsilon \in \mathbb{R}_{>0}$. Since $\lim_{x\to c} f(x) = L$, there is a $\delta \in \mathbb{R}_{>0}$ such that

$$x \in S \wedge 0 < |x-c| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

Since $\lim_{n\to\infty} s_n = c$, there is a $N \in \mathbb{N}$ such that

$$\begin{aligned} n \geq N &\Rightarrow |s_n - c| < \delta \\ &\Rightarrow |f(s_n) - L| < \epsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} f(s_n) = L$.

Conversely, suppose that $\lim_{x\to\infty c} f(x) \neq L$. Then, there exists an $\epsilon \in \mathbb{R}_{>0}$ such that for all $\delta \in \mathbb{R}_{>0}$ there is an $x \in S \setminus \{c\}$ such that $0 < |x-c| < \delta$ and $|f(x) - L| \geq \epsilon$. Therefore, for each $n \in \mathbb{N}$ there is a $s_n \in S \setminus \{c\}$ such that $0 < |s_n - c| < 1/n$ and $|f(s_n) - L| \geq \epsilon$. Hence, we have defined a sequence s such that $\operatorname{rng}(s) \subseteq S \setminus \{c\}$ and $\lim_{n\to\infty} s_n = c$ but $\lim_{n\to\infty} f(s_n) \neq L$. ☐

Using Theorem 2.1 we can apply everything we know about limits of sequences to limits of functions. For example, we have the following corollary.

Corollary 2.2. *Let $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$. Also, let $c \in S'$. Suppose that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to} g(x) = L'$. Then,*

(a) $\lim_{x\to c} (f(x) + g(x)) = L + L'$,

(b) $\lim_{x\to c} (k \cdot f(x)) = k \cdot L$, *for all* $k \in \mathbb{R}$,

(c) $\lim_{x\to c} (f(x) \cdot g(x)) = L \cdot L'$,

(d) $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{L'}$, *provided that* $L' \neq 0$.