Homework 5

Real Analysis

Due November 21, 2025

Exercises

1. Let $f(x) = x^2$ on $[0, 1]$. For each $n \in \mathbb{N}$, define $P = \{0, 1/n, 2/n, \ldots, 1\}$.
   (a) Find $L(f, P)$ and $U(f, P)$.
   (b) Show that f is Riemann integrable.
2. Let f and g be Riemann integrable on $[a, b]$. Prove that their product fg is Riemann integrable on $[a, b]$.
3. Let f and g be Riemann integrable on $[a, b]$. Suppose that $f(x) \leq g(x)$ for all $x \in [a, b]$. Prove that $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.
4. Let f be Riemann integrable on $[a, b]$. Prove that $|f|$ is Riemann integrable on $[a, b]$ and
$$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|\, dx.$$
5. Use part I of the fundamental theorem of calculus to find the derivative of each function.
   (a) $F(x) = \int_0^x \sqrt{1 + t^2}dt$
   (b) $F(x) = \int_0^{\sin(x)} \cos(t^2)dt$
   (c) $F(x) = \int_x^{x^2} \sqrt{1 + t^2}dt$
   (d) $F(x) = \int_0^x xe^{t^2}dt$, for $x \in [0, 1]$
6. Suppose that f and its first $(n+1)$ derivatives are continuous on a closed interval containing the point x_0. Use part II of the fundamental theorem of calculus and integration by parts to prove that for each $x \neq x_0$ in that interval, we have
$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}}{n!}(x - x_0)^n + R_n(x),$$
where
$$R_n(x) = \int_{x_0}^x \frac{(x - x_0)^n}{n!} f^{(n+1)}(t)dt.$$