Homework 4

Real Analysis

Due October 24, 2025

Exercises

1. Define $f\colon \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$
Use the $\epsilon - \delta$ definition to prove that $\lim_{x\to c} f(x)$ exists if and only if $c = 0$.

2. Define the modified Dirichlet function $f\colon (0,1) \to \mathbb{R}$ by
$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = m/n \text{ is a fully reduced rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$
Show that f is continuous at each irrational in $(0,1)$ and discontinuous at each rational.

3. Prove or give a counterexample: Every sequence of real numbers is a continuous function.

4. Suppose that $f\colon [a,b] \to \mathbb{R}$ is continuous and $f([a,b]) \subseteq \mathbb{Q}$. Prove that f is constant.

5. Prove or give a counterexample: If $f\colon A \to B$ is uniformly continuous on A and $g\colon B \to C$ is uniformly continuous on B, then $g \circ f\colon A \to C$ is uniformly continuous on A.

6. Let $S \subseteq \mathbb{R}$ and let $f\colon S \to \mathbb{R}$ be uniformly continuous on S. Prove that $f(S)$ is bounded.