Set Theory Homework

Real Analysis

Answer Key

1. **Set identities and Cartesian products.**

   Let A, B, and C be subsets of a universal set U.

   (a) Complete the proof of the following part of Theorem 2.3 (De Morgan's laws) from the notes:
   $$(A \cap B)^c = A^c \cup B^c.$$
   Begin with an arbitrary $x \in U$ and give a chain of logical equivalences that proves the equality.

   (b) Prove by double inclusion that
   $$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

   (c) Determine whether
   $$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$$
   holds for all sets A, B, and C. Prove the identity if it is true; otherwise, give a counterexample.

   Solution.

   (a) Let $x \in U$. Then
   $$\begin{aligned} x \in (A \cap B)^c &\iff x \notin A \cap B \\ &\iff x \notin A \text{ or } x \notin B \\ &\iff x \in A^c \text{ or } x \in B^c \\ &\iff x \in A^c \cup B^c. \end{aligned}$$
   Since this holds for every $x \in U$, the sets are equal.

   (b) If $(a, b) \in A \times (B \cap C)$, then $a \in A$ and $b \in B \cap C$. Hence $(a, b) \in A \times B$ and $(a, b) \in A \times C$, proving the forward inclusion. Conversely, if (a, b) lies in both $A \times B$ and $A \times C$, then $a \in A$, $b \in B$, and $b \in C$. Thus $(a, b) \in A \times (B \cap C)$.

   (c) The identity is true. An ordered pair (a, b) lies in the left side exactly when $a \in A$, $b \in B$, and $b \notin C$. This is equivalent to $(a, b) \in A \times B$ and $(a, b) \notin A \times C$, so it lies in the right side. The equivalence works in both directions.

2. **Equivalence relations.**

   Define a relation R on $\mathbb{Z}$ by
   $$mRn \iff 5 \mid (m - n).$$

(a) Prove that R is an equivalence relation. Address reflexivity, symmetry, and transitivity separately.

(b) Describe all distinct equivalence classes using set-builder notation.

(c) Explain why the distinct equivalence classes form a partition of $\mathbb{Z}$.

Solution. Reflexivity holds because $5 \mid (m-m) = 0$. If $5 \mid (m-n)$, then $5 \mid -(m-n) = n-m$, so R is symmetric. If $5 \mid (m-n)$ and $5 \mid (n-p)$, then 5 divides their sum $m-p$, so R is transitive. Thus R is an equivalence relation. Its distinct classes are

$$[r] = \{5k + r : k \in \mathbb{Z}\}, \qquad r = 0, 1, 2, 3, 4.$$

Every integer has exactly one remainder in $\{0, 1, 2, 3, 4\}$ upon division by 5, so the classes cover $\mathbb{Z}$ and are pairwise disjoint. Hence they form a partition of $\mathbb{Z}$.

3. **Relations and functions.**

Let $A = \{-2, -1, 0, 1, 2\}$ and $B = \{0, 1, 2, 3, 4\}$. Consider the relations from A to B defined by

$$R = \{(x, y) \in A \times B : y = x^2\}, \qquad S = \{(x, y) \in A \times B : y = x + 2\},$$

and

$$T = \{(x, y) \in A \times B : x = y^2\}.$$

(a) List the ordered pairs in each relation.

(b) Determine which of R, S, and T define functions from A to B. For each relation that fails to be a function, explain precisely which requirement in the definition of a function fails.

(c) For each relation that is a function, determine whether it is injective and whether it is surjective. Justify each answer.

(d) For each relation that is a bijective function, find its inverse function both as a set of ordered pairs and as a formula. Verify the two inverse identities using composition.

Solution. The relations are

$$\begin{aligned} R &= \{(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)\}, \\ S &= \{(-2, 0), (-1, 1), (0, 2), (1, 3), (2, 4)\}, \\ T &= \{(0, 0), (1, 1)\}. \end{aligned}$$

Both R and S are functions because every element of A occurs exactly once as a first coordinate. The relation T is not a function from A to B: the inputs $-2, -1$, and 2 have no outputs, so existence fails for those inputs. The function R is neither injective nor surjective, since $R(-1) = R(1)$ and its range is $\{0, 1, 4\}$. The function S is bijective. Its inverse is

$$S^{-1} = \{(0, -2), (1, -1), (2, 0), (3, 1), (4, 2)\}, \qquad S^{-1}(y) = y - 2.$$

For every $x \in A$ and $y \in B$, $S^{-1}(S(x)) = (x + 2) - 2 = x$ and $S(S^{-1}(y)) = (y - 2) + 2 = y$.

4. **Images and preimages.**

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 1$. Compute each of the following, and express your answers as intervals or unions of intervals when appropriate:

(a) $f([-2,1])$;

(b) $f^{-1}([0,3])$;

(c) $f^{-1}((-\infty,0))$.

Then let $E_1 = [-2,0]$ and $E_2 = [0,2]$. Compute

$$f(E_1 \cap E_2) \quad \text{and} \quad f(E_1) \cap f(E_2),$$

and explain why your result is consistent with the Theorem 5.6 from the notes on Set Theory. Finally, complete the proof of the following part of Theorem 5.6: for arbitrary $h : X \to Y$ and $F_1, F_2 \subseteq Y$,

$$h^{-1}(F_1 \cup F_2) = h^{-1}(F_1) \cup h^{-1}(F_2).$$

Solution. Since $f(x) = x^2 - 1$,

$$f([-2,1]) = [-1,3], \qquad f^{-1}([0,3]) = [-2,-1] \cup [1,2],$$

and

$$f^{-1}((-\infty,0)) = (-1,1).$$

Also, $E_1 \cap E_2 = \{0\}$, so $f(E_1 \cap E_2) = \{-1\}$. On the other hand, $f(E_1) = f(E_2) = [-1,3]$, and hence $f(E_1) \cap f(E_2) = [-1,3]$. This illustrates the possibly strict inclusion $f(E_1 \cap E_2) \subseteq f(E_1) \cap f(E_2)$.

For the proof, let $x \in X$. Then

$$\begin{aligned} x \in h^{-1}(F_1 \cup F_2) &\iff h(x) \in F_1 \cup F_2 \\ &\iff h(x) \in F_1 \text{ or } h(x) \in F_2 \\ &\iff x \in h^{-1}(F_1) \text{ or } x \in h^{-1}(F_2) \\ &\iff x \in h^{-1}(F_1) \cup h^{-1}(F_2). \end{aligned}$$

Thus the two sets are equal.

5. **Composition of functions.**

Define $f : [0,\infty) \to \mathbb{R}$ and $g : \mathbb{R} \to [0,\infty)$ by

$$f(x) = \sqrt{x}, \qquad g(x) = x^2.$$

(a) Find formulas for $g \circ f$ and $f \circ g$, including the domain and codomain of each composition. Are the two compositions equal?

(b) Determine whether f, g, and $g \circ f$ are injective and whether they are surjective. Justify each answer directly from the definitions.

(c) Use this example to explain why the converse of the statement "if f and g are injective, then $g \circ f$ is injective" need not hold.

(d) Prove directly that if $g \circ f$ is injective, then f is injective. Your proof should begin with arbitrary inputs x_1, x_2 in the domain of f satisfying $f(x_1) = f(x_2)$.

Solution. We have

$$(g \circ f)(x) = x \quad (x \in [0, \infty)), \qquad (f \circ g)(x) = \sqrt{x^2} = |x| \quad (x \in \mathbb{R}).$$

Thus $g \circ f : [0, \infty) \to [0, \infty)$ and $f \circ g : \mathbb{R} \to \mathbb{R}$; they are not equal (and do not even have the same domain and codomain). The function f is injective but not surjective onto $\mathbb{R}$. The function g is surjective onto $[0, \infty)$ but not injective. Their composition $g \circ f$ is the identity on $[0, \infty)$ and is therefore bijective. Thus $g \circ f$ can be injective even though g is not injective, so the stated converse fails.

In general, suppose $(g \circ f)$ is injective and $f(x_1) = f(x_2)$. Applying g gives $(g \circ f)(x_1) = (g \circ f)(x_2)$. Injectivity of $g \circ f$ now gives $x_1 = x_2$. Hence f is injective.