Fundamental Theorem of Calculus

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1 Preliminaries

The fundamental theorem of calculus is really two theorems, each expressing that differentiation and integration are inverse operations. Historically, the operations of integration and differentiation were developed ot solve seemingly unrelated problems. These problems may be described geometrically as finding the area under a curve and finding the slope of a curve at a point. The proof of their inverse relationship was one of the important theoretical (and practical) contributions of Newton and Leibniz in the seventeenth century.

Thus far, our definition of the Riemann integral $\int_a^b f(x)dx$ only holds when $a < b$. It will be convenient to extend this definition and let $\int_b^a f(x)dx = -\int_a^b f(x)dx$ and $\int_a^a f(x)dx = 0$. Also, our definition of differentiability only holds at interior points, so that both the left sided and right sided limits are defined. Here, it will be convenient to extend our definition of differentiability to include the boundary points of a closed interval.

2 The Fundamental Theorem of Calculus

The following theorem establishes the first part of the fundamental theorem of calculus.

Theorem 2.1. *Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable and define*

$$F(x) = \int_a^x f(t)dt,$$

for each $x \in [a,b]$. Then, F is uniformly continuous on $[a,b]$. Furthermore, if f is continuous at $c \in [a,b]$, then F is differentiable at c and $F'(c) = f(c)$.

Proof. Since f is Riemann integrable, it is bounded. So, there exists an $M \in \mathbb{R}_{>0}$ such that $|f(x)| \leq M$ for all $x \in [a,b]$. Let $\epsilon \in \mathbb{R}_{>0}$ and $\delta = \epsilon/M$. Also, note that, if $x \leq y$,

$$\begin{aligned}
|F(x) - F(y)| &= \left|\int_a^x f(t)dt - \int_a^y f(t)dt\right| \\
&= \left|-\int_x^y f(t)dt\right| \\
&\leq \int_x^y |f(t)|\,dt \leq M\,|x-y|
\end{aligned}$$

Also, if $x > y$,

$$\begin{aligned}
|F(x) - F(y)| &= \left|\int_a^x f(t)dt - \int_a^y f(t)dt\right| \\
&= \left|\int_y^x f(t)dt\right| \\
&\leq \int_y^x |f(t)dt| \leq M\,|x-y|
\end{aligned}$$

Hence, for all $x, y \in [a, b]$, if $|x - y| < \delta$ the

$$|F(x) - F(y)| < M|x - y| < M\delta = \epsilon.$$

Suppose that f is continuous at c. Also, note that, if $x \in [a, b]$ and $x < c$, then

$$\begin{aligned}
\left|\frac{F(x) - F(c)}{x - c} - f(c)\right| &= \left|\frac{1}{x - c}\left(\int_a^x f(t)dt - \int_a^c f(t)dt\right) - f(c)\right| \\
&= \left|\frac{1}{x - c}\int_c^x f(t)dt - \frac{1}{x - c}\int_c^x f(c)dt\right| \\
&= \left|\frac{1}{x - c}\int_c^x [f(t) - f(c)]dt\right| \leq \frac{1}{|x - c|}\int_c^x |f(t) - f(c)|\, dt.
\end{aligned}$$

Also, if $x \in [a, b]$ and $x > c$, then

$$\begin{aligned}
\left|\frac{F(x) - F(c)}{x - c} - f(c)\right| &= \left|\frac{1}{x - c}\left(\int_a^x f(t)dt - \int_a^c f(t)dt\right) - f(c)\right| \\
&= \left|\frac{1}{x - c}\int_c^x f(t)dt - \frac{1}{x - c}\int_c^x f(c)dt\right| \\
&= \left|\frac{1}{x - c}\int_c^x [f(t) - f(c)]dt\right| \leq \frac{1}{|x - c|}\int_c^x |f(t) - f(c)|\, dt.
\end{aligned}$$

Let $\epsilon \in \mathbb{R}_{>0}$. Since f is continuous at c, there is a $\delta \in \mathbb{R}_{>0}$ such that $|f(t) - f(c)| < \epsilon$ whenever $t \in N(c; \delta) \cap [a, b]$. Therefore, if $x \in N^*(c; \delta) \cap [a, b]$, then

$$\left|\frac{F(x) - F(c)}{x - c} - f(c)\right| \leq \frac{1}{|x - c|}\int_c^x |f(t) - f(c)|\, dt < \frac{1}{|x - c|}\int_c^x \epsilon dt = \epsilon.$$

□

The following theorem establishes the second part of the fundamental theorem of calculus.

Theorem 2.2. *Let $f\colon [a, b] \to \mathbb{R}$ be Riemann integrable and let $F\colon [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $F'(x) = f(x)$ for all $x \in (a, b)$, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof. Let P be any partition of $[a, b]$. For each $i \in \{1, 2, \ldots, n\}$, the mean value theorem states that there is a $t_i \in (x_{i-1}, x_i)$ such that

$$F'(t_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}},$$

that is, $f(t_i)\Delta x_i = F(x_i) - F(x_{i-1})$. Since $\sum_{i=1}^n f(t_i)\Delta x_i = \sum_{i=1}^n [F(x_i) - F(x_{i-1})] = F(b) - F(a)$,

$$L(f, P) \leq F(b) - F(a) \leq U(f, P).$$

Since the above bound holds for all partitions P of $[a, b]$,

$$L(f) \leq F(b) - F(a) \leq U(f).$$

Therefore, since f is Riemann integrable,

$$\int_a^b f(x)dx = L(f) = U(f) = F(b) - F(a).$$

□

3 Applications of The Fundamental Theorem of Calculus

Let $f\colon [a,b] \to \mathbb{R}$. If f is continuous on $[a,b]$, then Theorem 2.1 states that

$$F(x) = \int_a^x f(t)dt, \ \forall x \in [a,b]$$

is an *antiderivative* of f on $[a,b]$, that is, $F\colon [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ and $F'(x) = f(x)$ for all $x \in [a,b]$. Suppose $G(x)$ is also an antiderivative of $f(x)$ on $[a,b]$. Then, $H(x) = F(x) - G(x)$ is differentiable on $[a,b]$ and $H'(x) = 0$ for all $x \in [a,b]$. So, the mean value thoerem implies that $H(x)$ is constant. Therefore, every antiderivative of $f(x)$ on $[a,b]$ differs from $F(x)$ by a constant.

For example, let $f(x) = \sqrt{5+x^3}$ on $[0,1]$. Then, define $F(x) = \int_0^x \sqrt{5+t^3}dt$, for each $x \in [0,1]$. Since $f(x)$ is continuous on $[0,1]$, $F'(x) = \sqrt{5+x^3}$ for all $x \in [0,1]$.

Next, consider $f\colon [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x = m/n \text{ in lowest terms,} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then, define $F(x) = \int_0^x f(t)dt$ for each $x \in [0,1]$. By Theorem 2.1, $F(x)$ is uniformly continuous on $[0,1]$. Moreover, since $f(x)$ is continuous at all irrationals in $[0,1]$, $F'(x) = f(x) = 0$ for all irrational $x \in [0,1]$.

The following is a corollary of Theorem 2.1

Corollary 3.1. *Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$. Also, let $g\colon [c,d] \to \mathbb{R}$ be differentiable on $[c,d]$, where $g([c,d]) \subseteq [a,b]$. Define,*

$$F(x) = \int_a^{g(x)} f(t)dt,$$

for all $x \in [c,d]$. Then, $F'(x) = f(g(x))g'(x)$ for all $x \in [c,d]$.

Proof. Define $G(x) = \int_a^x f(t)dt$ for all $x \in [a,b]$, so that $F(x) = G(g(x))$ for all $x \in [c,d]$. Since f is continuous on $[a,b]$, f is Riemann integrable on $[a,b]$. Furthermore, by Theorem 2.1, $G'(x) = f(x)$ for all $x \in [a,b]$. Since g is differentiable on $[c,d]$ and $g([c,d]) \subseteq [a,b]$, the composition $F(x) = G(g(x))$ is differentiable on $[c,d]$ and the chain rule implies that

$$F'(x) = G'(g(x))g'(x),$$

for all $x \in [c,d]$. Therefore, $F'(x) = f(g(x))g'(x)$ for all $x \in [c,d]$. □

For example, let $f(x) = \sqrt{5+x^3}$ on $[0,1]$ and define

$$F(x) = \int_0^{x^2} \sqrt{5+t^3},$$

for all $x \in [0,1]$. Since f is continuous on $[0,1]$ and $g(x) = x^2$ is differentiable on $[0,1]$, where $g([0,1]) \subseteq [0,1]$, Corollary 3.1 implies that

$$F'(x) = f(g(x))g'(x) = 2x\sqrt{5+x^6},$$

for all $x \in [0,1]$.

The second part of the fundamental theorem of calculus is often used to evaluate definite integrals. For example, let $f(x) = x^3$ on $[1,4]$. Then, $F(x) = \frac{1}{4}x^4$ is an antiderivative of $f(x)$ on $[1,4]$. Therefore, Theorem 2.2 states that

$$\int_1^4 x^3 dx = F(4) - F(1) = \frac{1}{4}(256-1) = \frac{255}{4}.$$

The following proposition establishes the formula known as integration by parts.

Proposition 3.2. *Suppose that f and g are differentiable on $[a,b]$ and f' and g' are Riemann integrable on $[a,b]$. Then,*

$$\int_a^b f(x)g'(x)dx = [f(b)g(b) - f(a)g(a)] - \int_a^b f'(x)g(x)dx.$$

Proof. Define $h(x) = f(x)g(x)$ on $[a, b]$. Since both f and g are differentiable on $[a, b]$, so is h. Furthermore, by the product rule, $h'(x) = f'(x)g(x) + f(x)g'(x)$.

Since f and g are differentiable on $[a, b]$, they are Riemann integrable on $[a, b]$. Also, since f' and g' are assumed to be Riemann integrable, it follows that h' is Riemann integrable. Moreover, h is an antiderivative of h', so Theorem 2.2 states that

$$\int_a^b h'(x)dx = h(b) - h(a).$$

□