Exam III Worksheet

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Exercises

I. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is continuous at $x = 0$ but not differentiable at $x = 0$.

II. Let $f\colon [a,b] \to \mathbb{R}$. Prove that if f is differentiable at $c \in (a,b)$, then f is continuous at $c \in (a,b)$.

III. Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be differentiable at $c \in (a,b)$. Prove the following

(a) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

(b) $(f/g)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g^2(c)}$, if $g(c) \neq 0$.

IV. Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Complete the following

(a) State Rolle's Theorem.

(b) Use Rolle's theorem to prove the Cauchy Mean Value Theorem (see derivative worksheet 2).

(c) Show that the Cauchy Mean Value Theorem implies the Mean Value Theorem.

V. Let $f\colon [a,b] \to \mathbb{R}$ be bounded. State the definition of the following

(a) A partition P of $[a,b]$.

(b) The upper and lower Darboux sums of f with respect to P, denoted $U(f,P)$ and $L(f,P)$, respectively.

(c) The upper and lower Darboux integrals of f.

VI. Let $f(x) = x^3$ on $[0,1]$.

(a) Let $P = \{0, 1/n, 2/n, \ldots, 1\}$ be a partition of $[0,1]$ for each $n \in \mathbb{N}$.

(b) Find the upper and lower Darboux sums of f with respect to P.

(c) Show that

$$\frac{1}{4} \leq L(f) \leq U(f) \leq \frac{1}{4},$$

and conclude that $L(f) = U(f) = 1/4$.

VII. Let $f\colon [a,b] \to \mathbb{R}$ be bounded.

(a) State the definition of f being Riemann integrable.

(b) State the characterization of Rimeann integrable functions.

(c) Show that all monotone functions are Riemann integrable.

(d) Show that all continuous functions are Riemann integrable.

VIII. Let $f\colon [a,b] \to \mathbb{R}$ and $g\colon [a,b] \to \mathbb{R}$ be Riemann integrable.

(a) Show that kf is Riemann integrable for any $k \in \mathbb{R}$, and $\int_a^b kf = k\int_a^b f$

(b) Show that $f+g$ is Riemann integrable, and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

(c) Let $c \in (a,b)$. Show that $\int_a^b f = \int_a^c f + \int_c^b f$.

IX. Let $f\colon [a,b] \to \mathbb{R}$ be Riemann integrable.

(a) Prove the second part of the fundamental theorem of calculus: Suppose that $F\colon [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then, if $F'(x) = f(x)$ for all $x \in (a,b)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

(b) Prove the first part of the fundamental theorem of calculus: Define

$$F(x) = \int_a^x f(t)dt,$$

for each $x \in [a,b]$. Then, $F\colon [a,b] \to \mathbb{R}$ is uniformmly continuous on $[a,b]$. Moreover, if f is continuous at c, then F is differentiable at c and $F'(c) = f(c)$.

X. Use the fundametnal theorem of calculus to complete the following.

(a) Evaluate the following limit

$$\lim_{x\to 0} \frac{1}{x} \int_0^x \sqrt{9+t^2}dt.$$

(b) Let f be continuous on $[0,\infty)$. Suppose that $f(x) \neq 0$ for all $x > 0$ and that

$$f(x)^2 = 2\int_0^x f(t)dt,$$

for all $x \geq 0$. Prove that $f(x) = x$ for all $x \geq 0$.

(c) Let f be continuous on $[a,b]$. Suppose that

$$\int_a^x f(t)dt = \int_x^b f(t)dt,$$

for all $x \in [a,b]$. Prove that $f(x) = 0$ for all $x \in [a,b]$.