Exam II Worksheet

Thomas R. Cameron

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Exercises

III. Let $S \subseteq \mathbb{R}$ and let S' denote the accumulation points of S. Then, S' is closed.

Proof. Let x be an accumulation point of S' and let $\epsilon > 0$. Since x is an accumulation point of S', there is a $y \in N^*(x; \frac{\epsilon}{2}) \cap S'$. Since y is an accumulation point of S and $|x-y| > 0$, there is a $z \in N^*(y; |x-y|) \cap S$. Note that $z \in S$ and $z \neq x$. Furthermore,

$$\begin{aligned} |x-z| &= |x-y+y-z| \\ &\leq |x-y| + |y-z| \\ &< |x-y| + |x-y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

So, $z \in N^*(x;\epsilon) \cap S$ and it follows that x is an accumulation point of S. Therefore, $x \in S'$ and it follows that S' is closed. □

IV. (a) $[1,3) \subseteq \bigcup_{k=1}^{\infty} (0, 3-1/k)$. There is no finite subcover: for any fixed $k \in \mathbb{N}$, there is an $x \in [1,3)$ such that $x > 3 - 1/k$.

(b) $\mathbb{N} \subseteq \bigcup_{k=1}^{\infty} N(k;1)$. There is no finite subcover: for any fixed $k \in \mathbb{N}$, there is an $x \in \mathbb{N}$ such that $x > k+1$.

(c) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \subseteq \bigcup_{k=1}^{\infty}(1/k, 2)$. There is no finite subcover: for any fixed $k \in \mathbb{N}$, there is an $n \in \mathbb{N}$ such that $1/n < 1/k$.

VI. If $S \subseteq \mathbb{R}$ is infinite and bounded, then S has an accumulation point.

Proof. Suppose that S' is empty. Then, S is closed and every point in S is an isolated point. Since S is closed and bounded, the Heine-Borel theorem states that S is compact. Since every point in S is an isolated point, for each $x \in S$ there is a $\epsilon_x > 0$ such that $N(x;\epsilon_x) \cap S = \{x\}$. Then, $\mathcal{F} = \{N(x;\epsilon_x) : x \in S\}$ is an open cover of S. Since S is compact, there exists a finite subocver, that is, there exists a $n \in \mathbb{N}$ such that

$$S \subseteq \bigcup_{k=1}^{n} N(x_k; \epsilon_{x_k}).$$

However, each neighborhood contains exactly one element of S, which contradicts S being infinite. Therefore, S' must be non-empty. □

VII. (a) The sequence $x(n) = \frac{(-1)^n}{n}$ is Cauchy but not monotone.

(b) The sequence $x(n) = n$ is monotone but not Cauchy.

(c) The sequence $x(n) = (-1)^n$ is bounded but not Cauchy.

IX. Let $f : [a,b] \to \mathbb{R}$ be continuous. Then, for each y between $f(a)$ and $f(b)$, there exists a $c \in (a,b)$ such that $f(c) = y$

Proof. Suppose that $f(a) < f(b)$ and let $y \in (f(a), f(b))$. Define $g(x) = f(x) - y$; note that $g(x)$ is continuous on $[a, b]$. Then, $g(a) < 0 < g(b)$. Therefore, problem VIII states that there exists a $c \in (a, b)$ such that $g(c) = 0$. Hence, $f(c) = y$.

Suppose that $f(a) > f(b)$ and let $y \in (f(b), f(a))$. Define $g(x) = y - f(x)$; note that $g(x)$ is continuous on $[a, b]$ Then, $g(a) < 0 < g(b)$ Therefore, Problem VIII states that there exists a $c \in (a, b)$ such that $g(c) = 0$. Hence, $f(c) = y$. $\square$