

Exam I Worksheet Solutions

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Exercises

III. Let $f: A \rightarrow B$ and suppose there exists a $g: B \rightarrow A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.

(a) f is injective.

Proof. Suppose that $(a, b), (a', b) \in f$. Since f is a function, it follows that the b related to a and a' by f must be unique. Furthermore, since $(a, a), (a', a') \in i_A$ and

$$i_A = g \circ f = \{(a, a): \exists b \in B \ni (a, b) \in f \wedge (b, a) \in g\},$$

it follows that $(b, a), (b, a') \in g$. Since g is a function, $a = a'$. Therefore, f is injective. $\square$

(b) f is surjective.

Proof. Let $b \in B$. Then, $(b, b) \in i_B$. Since

$$i_B = f \circ g = \{(b, b): \exists a \in A \ni (b, a) \in g \wedge (a, b) \in f\},$$

it follows that there exists an $a \in A$ such that $(a, b) \in f$. Therefore, f is surjective. $\square$

(c) $g = f^{-1}$.

Proof. Since $f: A \rightarrow B$ is bijective, it follows that $f^{-1}: B \rightarrow A$ is also a bijective function. Therefore, f^{-1} and g are functions with the same domain, that is, $\text{dom}(g) = \text{dom}(f^{-1}) = B$. Now, let $b \in B$. Then, $(b, b) \in i_B$. Furthermore, since $i_B = f \circ g$ and $i_B = f \circ f^{-1}$, it follows that there exists an $a \in A$ such that $(b, a) \in g$ and $(b, a) \in f^{-1}$. $\square$

IV. Let $r \neq 1$. Then, for each $n \in \mathbb{N}$,

$$\sum_{i=0}^n r^i = \frac{1 - r^{n+1}}{1 - r}.$$

Proof. We proceed via mathematical induction. For the base case, $n = 1$, note that

$$1 + r = \frac{1 - r^2}{1 - r}.$$

Let $k \in \mathbb{N}$ and suppose that

$$\sum_{i=0}^k r^i = \frac{1 - r^{k+1}}{1 - r}.$$

Then,

$$\begin{aligned}
\sum_{i=0}^{k+1} r^i &= \sum_{i=0}^k r^i + r^{k+1} \\
&= \frac{1 - r^{k+1}}{1 - r} + r^{k+1} \\
&= \frac{1 - r^{k+1}}{1 - r} + r^{k+1} \cdot \frac{1 - r}{1 - r} \\
&= \frac{1 - r^{k+2}}{1 - r}.
\end{aligned}$$

Therefore, $\sum_{i=0}^n r^i = \frac{1-r^{n+1}}{1-r}$ for all $n \in \mathbb{N}$. □

VI. Let $x: \mathbb{N} \rightarrow \mathbb{Q}$ and $y: \mathbb{N} \rightarrow \mathbb{Q}$ denote rational Cauchy sequences. Then, $x + y = (x_n + y_n)_{n=1}^\infty$ and $x \cdot y = (x_n \cdot y_n)_{n=1}^\infty$ are rational Cauchy sequences.

Proof. The fact that $x + y$ and $x \cdot y$ are rational follows from the closure of $\mathbb{Q}$. Hence, we only need to show that these sequences are Cauchy. To this end, let $\epsilon > 0$. Then, since x and y are Cauchy, there exists $N_1, N_2 \in \mathbb{N}$ such that for all $n, m \geq N_1$,

$$|x_n - x_m| < \frac{\epsilon}{2}$$

and for all $n, m \geq N_2$,

$$|y_n - y_m| < \frac{\epsilon}{2}.$$

Therefore, for $n, m \geq \max\{N_1, N_2\}$,

$$\begin{aligned}
|(x_n + y_n) - (x_m + y_m)| &= |(x_n - x_m) + (y_n - y_m)| \\
&\leq |x_n - x_m| + |y_n - y_m| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Since x and y are Cauchy, they are both bounded sequences. That is, there exists $M_x, M_y \in \mathbb{Q}_{>0}$ such that $|x_n| \leq M_x$ and $|y_n| \leq M_y$, for all $n \in \mathbb{N}$. Furthermore, there exists $N_1, N_2 \in \mathbb{N}$ such that for all $n, m \geq N_1$,

$$|x_n - x_m| < \frac{\epsilon}{2M_y}$$

and for all $n, m \geq N_2$,

$$|y_n - y_m| < \frac{\epsilon}{2M_x}.$$

Therefore, for $n, m \geq \max\{N_1, N_2\}$,

$$\begin{aligned}
|x_n \cdot y_n - x_m \cdot y_m| &= |x_n \cdot y_n - x_m \cdot y_n + x_m \cdot y_n - x_m \cdot y_m| \\
&= |(x_n - x_m)y_n + x_m(y_n - y_m)| \\
&\leq |y_n| |x_n - x_m| + |x_m| |y_n - y_m| \\
&\leq M_y |x_n - x_m| + M_x |y_n - y_m| \\
&< M_y \frac{\epsilon}{2M_y} + M_x \frac{\epsilon}{2M_x} = \epsilon.
\end{aligned}$$

□

VII. Let $\mathcal{C}$ denote the set of all rational Cauchy sequences. Define the relation R on $\mathcal{C}$ by

$$((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty) \in R \Leftrightarrow \forall \epsilon \in \mathbb{Q}_{>0}, \exists N \in \mathbb{N} \ni n \geq N \Rightarrow |x_n - y_n| < \epsilon.$$

Then R is an equivalence relation on $\mathcal{C}$.

Proof. Let $(x_n)_{n=1}^\infty \in \mathcal{C}$. Then, for any $\epsilon \in \mathbb{Q}_{>0}$, it follows that $|x_n - x_n| = 0 < \epsilon$ for all $n \in \mathbb{N}$. Hence, the relation R is reflexive. Suppose that $((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty) \in R$. Then, for all $\epsilon \in \mathbb{Q}_{>0}$ there exists a $N \in \mathbb{N}$ such that $|x_n - y_n| < \epsilon$. Therefore,

$$|y_n - x_n| = |x_n - y_n| < \epsilon,$$

thus, $((y_n)_{n=1}^\infty, (x_n)_{n=1}^\infty) \in R$. Hence, the relation R is symmetric. Suppose that $((x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty) \in R$ and $((y_n)_{n=1}^\infty, (z_n)_{n=1}^\infty) \in R$. Then, for all $\epsilon \in \mathbb{Q}_{>0}$ there exists a $N_1, N_2 \in \mathbb{N}$ such that $|x_n - y_n| < \epsilon/2$ and $|y_n - z_n| < \epsilon/2$ for all $n \geq \max\{N_1, N_2\}$. Therefore,

$$\begin{aligned} |x_n - z_n| &= |x_n - y_n + y_n - z_n| \\ &\leq |x_n - y_n| + |y_n - z_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Hence, the relation R is transitive. □

X. The natural numbers are unbounded above in $\mathbb{R}$.

Proof. Suppose that $\mathbb{N} \subseteq \mathbb{R}$ is bounded above. By the completeness of the real numbers, the least upper bound of $\mathbb{N}$ is a real number, that is, $L = \sup \mathbb{N} \in \mathbb{R}$. Since $L - 1$ is not an upper bound of $\mathbb{N}$, there exists a $n \in \mathbb{N}$ such that $n > L - 1$. However, $n + 1 \in \mathbb{N}$ and $n + 1 > L$, which contradicts L being an upperbound on $\mathbb{N}$. □