Exam I Worksheet

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Exercises

I. Let p and q be logical statements and let c be a statement that is always false. Use a truth table to show that $p \Rightarrow q$ is logically equivalent to $(p \wedge \neg q) \Rightarrow c$.

II. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$ and define the functions

$$f = \{(1, a), (2, b), (3, a)\} \text{ and } g = \{(a, 1), (b, 3), (c, 2)\}.$$

Describe each of the following functions by listing its ordered pairs. State the domain and range of each function and whether they are injective or surjective.

(a) g^{-1}

(b) $f \circ g$

(c) $g \circ f$

III. Let $f \colon A \to B$ and suppose there exists a $g \colon B \to A$ such that $g \circ f = i_A$ and $f \circ g = i_B$.

(a) Prove that f is injective.

(b) Prove that f is surjective.

(c) Prove that $g = f^{-1}$.

IV. Let $r \neq 1$ be a real number. Use the principal of mathematical induction to prove that

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r},$$

for all $n \in \mathbb{N}$.

V. Let $s \colon \mathbb{N} \to \mathbb{Q}$ denote a rational Cauchy sequence. Prove that there exists a rational $M \in \mathbb{Q}$ such that $|s_n| \leq M$ for all $n \in \mathbb{N}$.

VI. Let $x \colon \mathbb{N} \to \mathbb{Q}$ and $y \colon \mathbb{N} \to \mathbb{Q}$ denote rational Cauchy sequences. Prove that $x + y = (x_n + y_n)_{n=1}^{\infty}$ and $x \cdot y = (x_n \cdot y_n)_{n=1}^{\infty}$ are rational Cauchy sequences.

VII. Let $\mathcal{C}$ denote the set of all rational Cauchy sequences. Define the relation R on $\mathcal{C}$ by

$$((x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}) \in R \iff \forall \epsilon \in \mathbb{Q}_{>0},\ \exists N \in \mathbb{N} \ni n \geq N \Rightarrow |x_n - y_n| < \epsilon.$$

Show that R is an equivalence relation.

VIII. State what it means for the real numbers $\mathbb{R}$ to be a complete ordered field.

IX. For each of the following subsets of $\mathbb{R}$, give its infimum and supremum if they exist.

(a) $\{1, 3\}$

(b) $(-\infty, 4)$

(c) $\left\{\frac{n}{n+1} : n \in \mathbb{N}\right\}$

X. Use the completeness of $\mathbb{R}$ to prove that the set of natural numbers $\mathbb{N}$ are unbounded above in $\mathbb{R}$.