Continuous Functions

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1 Continuous Functions

Let $f\colon S \to \mathbb{R}$. We've seen that the limit of a function f at a point $c \in S'$ is independent of the nature of the function at c. It may be that f is not defined at c, and even if $f(c)$ exists it may differ from the value of the limit. When it happens that the limit of f at c is equal to $f(c)$, the function is said to be continuous at c. More formally, we say that f is *continuous* at $c \in S$ if for all $\epsilon \in \mathbb{R}_{>0}$ there is a $\delta \in \mathbb{R}_{>0}$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$ and $x \in S$.

Note that the definition of continuity does not require c to be an accumulation point of S. If c is an isolated point of S, then the definition of continuity follows immediately. If c is an accumulation point of S, then f is continuous at c if and only if f has a limit at c and $\lim_{x\to c} f(x) = f(c)$. The following result summarizes equivalent conditions for the continuity of f at c.

Theorem 1.1. *Let $f\colon S \to \mathbb{R}$ and let $c \in S$. Then, the following three conditions are equivalent:*

(a) f is continuous at c.

(b) If $x\colon \mathbb{N} \to S$ converges to c, then $\lim_{n\to\infty} f(x_n) = f(c)$.

(c) For every neighborhood V of $f(c)$ there is a neighborhood U of c such that $f(U \cap S) \subseteq V$.

Furthermore, if c is an accumulation point of S, then the above are all equivalent to

(d) f has a limit at c and $\lim_{x\to c} f(x) = f(c)$.

Proof. Let c be an isolated point of S. Then, there exists a $\delta \in \mathbb{R}_{>0}$ such that $N(c;\delta) \cap S = \{c\}$. Hence, $f(N(c;\delta) \cap S) = \{f(c)\} \subseteq V$ for any neighborhood V of $f(c)$, that is, (c) holds. If $x \in S$ and $|x - c| < \delta$, then $x = c$; so, $|f(x) - f(c)| = 0 < \epsilon$ for all $\epsilon \in \mathbb{R}_{>0}$. Hence, f is continuous at c, that is, (a) holds. If $x\colon \mathbb{N} \to S$ converges to c, then there is a $N \in \mathbb{N}$ such that $|x_n - c| < \delta$ for all $n \geq N$; so, $x_n = c$ for all $n \geq N$. Hence, $f(x_n) = f(c)$ for all $n \geq N$, so $\lim_{n\to\infty} f(x_n) = f(c)$, that is, (b) holds. It follows that if c is an isolated point of S, then (a), (b), and (c) all hold true.

Let c be an accumulation point of S. Suppose that (a) holds and $x\colon \mathbb{N} \to S$ converges to c. Since f is continuous at c, for any $\epsilon \in \mathbb{R}_{>0}$, there is a $\delta \in \mathbb{R}_{>0}$ such that $|f(x) - f(c)| < \epsilon$ whenever $x \in N(c;\delta) \cap S$. Since $x\colon \mathbb{N} \to S$ converges to c, there is a $N \in \mathbb{N}$ such that $|x_n - c| < \delta$ whenever $n \geq N$. Therefore,

$$\begin{aligned} n \geq N &\Rightarrow |x_n - c| < \delta \\ &\Rightarrow |f(x_n) - f(c)| < \epsilon. \end{aligned}$$

Hence, $\lim_{n\to\infty} f(x_n) = f(c)$, and it follows that (a) implies (b).

To show that (b) implies (c), we establish the converse. To that end, suppose there exists an $\epsilon \in \mathbb{R}_{>0}$ such that for all $\delta \in \mathbb{R}_{>0}$ there exists $x \in N(c;\delta) \cap S$ such that $f(x) \notin N(f(c);\epsilon)$. Then, for each $n \in \mathbb{N}$, there exists an $x_n \in N(c;1/n) \cap S$ such that $f(x_n) \notin N(f(c);\epsilon)$. Therefore, $x\colon \mathbb{N} \to S$ converges to c but $\lim_{n\to\infty} f(x_n) \neq f(c)$.

Suppose that (c) holds. Let $\epsilon \in \mathbb{R}_{>0}$. Then, there exists a $\delta \in \mathbb{R}_{>0}$ such that $f(x) \in N(f(c);\epsilon)$ whenever $x \in N(c;\delta) \cap S$. Since c is an accumulation point of S, there are points in $N(c;\delta) \cap S$ other than c itself. Therefore, f has a limit at c and $\lim_{x\to c} f(x) = f(c)$. So, (c) implies (d).

Suppose (d) holds. Let $\epsilon \in \mathbb{R}_{>0}$. Then, there exists a $\delta \in \mathbb{R}_{>0}$ such that $f(x) \in N(f(c);\epsilon)$ whenever $x \in N^*(c;\delta)$. Clearly $f(c) \in N(f(c);\epsilon)$. Therefore, $f(x) \in N(f(c);\epsilon)$ whenever $x \in N(c;\delta)$; thus, f is continuous at c. So, (d) implies (a). □

Theorem 1.1 can be useful to show a function is discontinuous. For example, consider the Dirichlet function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Let $c \in \mathbb{R}$ and $\epsilon = 1/2$. Then, for all $\delta \in \mathbb{R}_{>0}$, the neighborhood $N(c;\delta)$ contains both rational and irrational numbers. Therefore, $f(N(c;\delta))$ is not a subset of $N(f(c);\epsilon)$ for any $\delta \in \mathbb{R}_{>0}$. So, Theorem 1.1 (c) implies that f is not coninuous at c.

Therefore 1.1 also implies that the properties of limits can be used to identify properties of continuous functions. The following corollary summarizes these properties.

Corollary 1.2. *Let $f\colon S \to \mathbb{R}$ and $g\colon S \to \mathbb{R}$ be continuous at c. Then,*

(a) $f + g$ is continuous at c.

(b) $k \cdot f$ is continuous at c, for all $k \in \mathbb{R}$.

(c) $f \cdot g$ is continuous at c.

(d) f/g is continuous at c, provided that $g(c) \neq 0$.

In addition to Corollary 1.2, the following theorem shows that the composition of continuous functions is continuous.

Theorem 1.3. *Let $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ such that $f(A) \subseteq B$. Suppose that f is continuous at $c \in A$ and g is continuous at $f(c) \in B$. Then, $g \circ f$ is continuous at c.*

Proof. Let W be any neighborhood of $g(f(c))$. Since g is continuous at $f(c)$, there exists a neighborhood V of $f(c)$ such that $g(V \cap B) \subseteq W$. Since f is continuous at c, there is a neighborhood U of c such that $f(U \cap A) \subseteq V$. Since $f(A) \subseteq B$, it follows that $f(U \cap A) \subseteq V \cap B$. Therefore,

$$g(f(U \cap A)) \subseteq g(V \cap B) \subseteq W.$$

Hence, Theorem 1.1 (c) implies that $g \circ f$ is continuous at c. □

2 Compactness and Continuous Functions

In this section, we show that for continuous functions the image of a compact set is another compact set. Note that $f\colon S \to \mathbb{R}$ is continuous on S if $f(c)$ is continuous for all $c \in S$. We begin with the following lemma that establishes that the image is bounded.

Lemma 2.1. *Let $S \subseteq \mathbb{R}$ and $f\colon S \to \mathbb{R}$ be continuous on S. If S is compact, then $f(S)$ is bounded.*

Proof. For the sake of contradiction, suppose that S is compact and $f(S)$ is unbounded. Then, for each $n \in \mathbb{N}$, there is a $s_n \in S$ such that $|f(s_n)| \geq n$. The sequence $s\colon \mathbb{N} \to S$ is bounded since S is compact. Therefore, the Bolzano-Weierstrass theorem implies that $\operatorname{rng}(s)$ has an accumulation point, which we denote by $L \in R$. Since $\operatorname{rng}(s) \subseteq S$, it follows that L is an accumulation point of S; hence, since S is compact, $L \in S$. Furthermore, since L is an accumulation point of $\operatorname{rng}(s)$, there exists a subsequence $s \circ \sigma$ that converges to L. However, since $f(s_{\sigma_k}) \geq \sigma_k \geq k$, it follows that $\lim_{k\to\infty} f(s_{\sigma_k})$ does not exist, which contradicts the continuity of f at L by Theorem 1.1 (b). □

Next, we show that the image is compact.

Theorem 2.2. *Let $S \subseteq \mathbb{R}$ and $f\colon S \to \mathbb{R}$ be continuous on S. If S is compact, then $f(S)$ is compact.*

Proof. By Lemma 2.1, $f(S)$ is bounded. Thus, by the Heine-Borel theorem, we only need to show that $f(S)$ is closed. If $f(S)$ has no accumulation points, then we are done. Suppose that $b \in \mathbb{R}$ is an accumulation point of $f(S)$. Then, for each $n \in \mathbb{N}$, there is a $y_n \in N^*(b;1/n) \cap f(S)$ and $x_n \in S$ such that $f(x_n) = y_n$. Note that the y_n can be selected to be distinct since there are infinitely many points in $N^*(b;1/n) \cap f(S)$, for all

$n \in \mathbb{N}$. Therefore, $\operatorname{rng}(y)$ and $\operatorname{rng}(x)$ are infinite sets. Since $\operatorname{rng}(x) \subseteq S$ is bounded, the Bolzano-Weierstrass theorem states that $\operatorname{rng}(x)$ has an accumulation point, which we denote by $a \in R$. Since $\operatorname{rng}(x) \subseteq S$, it follows that a is an accumulation point of S; hence, since S is compact, $a \in S$. Furthermore, since a is an accumulation point of $\operatorname{rng}(x)$, there exists a subsequence $x \circ \sigma$ that converges to a. Since f is continuous, Theorem 1.1 (b) implies that

$$f(a) = \lim_{k \to \infty} f(x_{\sigma_k}) = \lim_{k \to \infty} y_{\sigma_k} = b.$$

Since $f(a) = b$, it follows that $b \in f(S)$. Therefore, $f(S)$ is closed. □

As a corollary of Theorem 2.2, we show that the image contains its maximum and minimum value.

Corollary 2.3. *Let $S \subseteq \mathbb{R}$ and $f\colon S \to \mathbb{R}$ be continuous on S. If S is compact, then $f(S)$ contains its maximum and minimum value.*

Proof. Suppose that S is compact. Then, Theorem 2.2 states that $f(S)$ is compact. Therefore, $f(S) \subseteq \mathbb{R}$ is bounded; so, the completeness axiom states that $f(S)$ has an infimum and supremum. Let m denote the supremum of $f(S)$ and for the sake of contradiction suppose that $m \notin f(S)$. Then, for all $\epsilon > 0$, there is a $y \in f(S)$ such that $m - \epsilon < y < m$. Therefore, m is an accumulation point of $f(S)$. Furthermore, $m \notin f(S)$ contradicts $f(S)$ being closed. A similar argument shows that $f(S)$ contains its infimum. Thus, $f(S)$ contains its maximum and minimum value. □