Completeness

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1 Suprema and Infema

We've seen how the real numbers can be constructed as equivalence classes of rational Cauchy sequences. Using this construction, we are able to show that the real numbers are an ordered field and a metric space. However, these properties also hold for the rational numbers. What makes the real numbers special is that they include the irrational numbers, which leads to a property known as completeness.

Let $S \subseteq \mathbb{R}$ be non-empty. If there exists a $m \in \mathbb{R}$ such that $m \geq s$ for all $s \in S$, then m is an *upper bound* for S. If $m \in \mathbb{R}$ satisfies $m \leq s$ for all $s \in S$, then m is a *lower bound* for S. If S is bounded above, then its least upper bound is called its *supremum*, denoted $\sup S$. In particular, $m = \sup S$ provided that

(a) For all $s \in S$, $m \geq s$.

(b) If $m' < m$, there exists a $s' \in S$ such that $s' > m$.

If S is bounded below, then its greatest lower bound is called its *infimum*, denoted $\inf S$. In particular, $m = \inf S$ provided that

(a) For all $s \in S$, $m \leq s$.

(b) If $m' > m$, there exists a $s' \in S$ such that $s' < m$.

The completeness of $\mathbb{R}$ guarantees that every non-empty subset of real numbers that is bounded above has a least upper bound in $\mathbb{R}$. This property does not hold for the rational numbers, for example, the set

$$T = \left\{q \in \mathbb{Q} \colon 0 \leq q^2 \leq 2\right\}$$

does not have a supremum in $\mathbb{Q}$. The following result proves the completness of $\mathbb{R}$, note that this proof relies heavily on the density of the rationals in the reals.

Theorem 1.1. *Let $S \subseteq \mathbb{R}$ be non-empty and bounded above. Then,* $\sup S \in \mathbb{R}$.

Proof. Let $s \in S$ and let m be an upper bound of S. By the density of the rationals, there exists $a_1, b_1 \in \mathbb{Q}$ such that $a_1 < s$ and $b_1 > m$. We define the rational sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ recursively as follows. Given a_n, b_n define

$$m_n = \frac{a_n + b_n}{2}.$$

If m_n is an upper bound of S, then define $a_{n+1} = a_n$ and $b_{n+1} = m_n$. Otherwise, there exists an $x \in S$ such that $x > m_n$. By the density of the rationals, there exists a $q \in \mathbb{Q}$ such that $m_n < q < x$. In this case, define $a_{n+1} = q$ and $b_{n+1} = b_n$.

Note that a_n is never an upper bound of S, while b_n is always an upper bound of S. Furthermore,

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$$

and

$$|b_n - a_n| \leq \frac{|b_1 - a_1|}{2^{n-1}},$$

for all $n \geq 1$.

Therefore, the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are Cauchy. Indeed, let $\epsilon > 0$ and select $N \in \mathbb{N}$ so that $\frac{|b_1 - a_1|}{2^{N-1}} < \epsilon$. Then, for all $m, n \geq N$

$$|a_n - a_m| \leq |b_m - a_m| \leq \frac{|b_1 - a_1|}{2^{n-1}} < \epsilon$$

and

$$|b_n - b_m| \leq |b_n - a_n| \leq \frac{|b_1 - a_1|}{2^{n-1}} < \epsilon.$$

Moreover, these sequences are members of the same equivalence class since their difference converges to zero. Let α denote the real number corresponding to the equivalence class containing the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$. We claim that α is the least upper bound of S.

If α were not an upper bound of S, then there exists $x \in S$ such that $\alpha < x$. By the density of the rationals, there exists a $q \in \mathbb{Q}$ such that $\alpha < q < x$. Therefore, by definition of the ordering on the reals, there exists a rational $\epsilon > 0$ and $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow b_n < q - \epsilon.$$

However, each b_n is an upper bound of S, so $b_n \geq x > q$ for all $n \in \mathbb{N}$, which leads to the contradiction $(q - \epsilon) > q$. Hence, α must be an upper bound of S.

In addition, if α were not a least upper bound, then there exists a $\beta \in \mathbb{R}$ such that $\beta < \alpha$ and β is an upper bound of S. By density of the rationals, there exists a $q \in \mathbb{Q}$ such that $\beta < q < \alpha$. Therefore, by definition of the ordering on the reals, there exists a rational $\epsilon > 0$ and $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow a_n > q + \epsilon.$$

However, no a_n is an upper bound of S, so for all $x \in S$, $a_n < x < q$, which leads to the contradiction $q > (q + \epsilon)$. Hence, α must be the least upper bound of S. $\square$