

Compactness Worksheet

Real Analysis

Answer Key

1. Closed subsets of compact sets.

Let $K \subseteq \mathbb{R}$ be compact, and let $F \subseteq K$ be closed in $\mathbb{R}$. Prove that F is compact by completing the following steps.

- (a) Let $\mathcal{U}$ be an arbitrary open cover of F . Why can the compactness of K not yet be applied directly to $\mathcal{U}$?
- (b) Since F is closed, identify one open set that can be added to $\mathcal{U}$ so that the enlarged collection covers K . Prove that it covers K .
- (c) Apply compactness of K to the enlarged cover. Explain why the additional set may be removed, if it was selected, while the remaining sets still cover F . Conclude that F is compact, and identify where closedness of F and compactness of K were used.

The additional open set should cover the points of K that do not belong to F .

Solution. The collection $\mathcal{U}$ need not cover K , so compactness of K does not yet apply. Because F is closed, F^c is open. The collection $\mathcal{U} \cup \{F^c\}$ covers K : points in F are covered by $\mathcal{U}$, while points in $K \setminus F$ lie in F^c . Compactness of K supplies a finite subcover. If F^c occurs, remove it. No point of F is lost because $F \cap F^c = \emptyset$. The remaining finite subcollection is selected from $\mathcal{U}$ and covers F , so F is compact. Closedness makes F^c open, and compactness produces the finite selection.

2. Completing the Heine–Borel theorem.

For $S \subseteq \mathbb{R}$, prove that

$$S \text{ is compact} \iff S \text{ is closed and bounded.}$$

- (a) Assume that S is compact. State the two earlier propositions that show that S is closed and bounded.
- (b) Conversely, suppose that S is closed and bounded. Explain why there is an $M > 0$ such that $S \subseteq [-M, M]$. Which theorem shows that $[-M, M]$ is compact? Apply Question 1 to finish the proof.
- (c) Explain why both conditions are necessary by giving a bounded set that is not closed and a closed set that is not bounded. Why is neither set compact?

The reverse direction should now follow by combining the compact interval theorem with Question 1.

Solution. If S is compact, the earlier propositions that compact subsets of $\mathbb{R}$ are bounded and that compact subsets of $\mathbb{R}$ are closed give the two required conclusions. Conversely, boundedness gives an $M > 0$ such that $|x| \leq M$ for every $x \in S$; hence $S \subseteq [-M, M]$. The compact interval theorem says that $[-M, M]$ is compact. Since S is a closed subset of this compact interval, Question 1 implies that S is compact. For necessity, $(0, 1)$ is bounded but not closed, while $\mathbb{Z}$ is closed but unbounded. Neither is compact by the forward direction of the theorem.

3. Extreme points of a compact set.

Let $K \subseteq \mathbb{R}$ be nonempty and compact. Prove that K has a maximum and a minimum.

- (a) Use Heine–Borel and completeness of $\mathbb{R}$ to explain why $s = \sup K$ exists.
- (b) Let $\varepsilon > 0$. Why is $s - \varepsilon$ not an upper bound for K ? Deduce that some $x \in K$ satisfies

$$s - \varepsilon < x \leq s.$$

- (c) Use part (b) to show that every neighborhood of s intersects K . Why does closedness of K imply that $s \in K$? Conclude that $s = \max K$, and briefly explain how the analogous argument using $\inf K$ proves that K has a minimum.

Compactness supplies boundedness so that the supremum exists, and closedness so that the supremum remains in the set.

Solution. Heine–Borel implies that K is bounded. Since K is also nonempty, completeness gives $s = \sup K$. For $\varepsilon > 0$, the number $s - \varepsilon$ cannot be an upper bound; otherwise it would be an upper bound smaller than the least upper bound s . Thus some $x \in K$ satisfies $s - \varepsilon < x \leq s$, so $x \in N(s; \varepsilon) \cap K$. Every neighborhood of s therefore meets K , and $s \in \text{cl}(K)$. Heine–Borel also implies that K is closed, so $s \in K$ and $s = \max K$. For the minimum, boundedness below and nonemptiness give $m = \inf K$. For every $\varepsilon > 0$, $m + \varepsilon$ is not a lower bound, so some $y \in K$ satisfies $m \leq y < m + \varepsilon$. Thus every neighborhood of m meets K . Closedness implies $m \in K$, and therefore $m = \min K$.

4. At-home continuation: examples and extreme points.

For each set below, decide whether it is closed, bounded, and compact. When the set is nonempty, find its supremum and infimum and determine whether it has a maximum and a minimum.

$$\begin{aligned} A &= \{0\} \cup \{1/n : n \in \mathbb{N}\}, & B &= \{1/n : n \in \mathbb{N}\}, \\ C &= \{x \in \mathbb{R} : x^2 \leq 5\}, & D &= [0, 1] \cup (2, 3]. \end{aligned}$$

- (a) Which sets show that boundedness alone does not imply compactness?
- (b) Which set shows that a noncompact set may nevertheless have both a maximum and a minimum?
- (c) Compare A and B . How does including a single accumulation point change the compactness conclusion?

Use Heine–Borel for compactness and Question 3 as a consistency check for the extreme points of compact sets.

Solution. The set A is bounded and contains its only accumulation point, 0, so it is closed and compact. Moreover, $\sup A = \max A = 1$ and $\inf A = \min A = 0$. The set B is bounded but not closed because $0 \in B' \setminus B$, so it is not compact. It has $\sup B = \max B = 1$ and $\inf B = 0$, but no minimum. The set $C = [-\sqrt{5}, \sqrt{5}]$ is closed, bounded, and compact, with maximum $\sqrt{5}$ and minimum $-\sqrt{5}$. The set D is bounded but not closed because $2 \in D' \setminus D$, so it is not compact. Nevertheless, $\inf D = \min D = 0$ and $\sup D = \max D = 3$. Thus B and D show that boundedness is insufficient, and D shows that a noncompact set may have both extreme points. Adding the missing accumulation point 0 to B produces the compact set A .