Compactness

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1 Compactness

Compactness is a topological generalization of finiteness that allows us to extend results surrounding functions over finite subsets of real numbers to functions over compact subsets of real numbers. A set S is *compact* if whenever it is contained in the union of a family $\mathcal{F}$ of open sets, then it is contained in the union of some finite number of sets in $\mathcal{F}$. If $\mathcal{F}$ is a family of open sets whose union contains S, then $\mathcal{F}$ is called an *open cover* of S. If $\mathcal{G} \subseteq \mathcal{F}$ and $\mathcal{G}$ is also an open cover of S, then $\mathcal{G}$ is called a *subcover* of S. Thus, S is compact if and only if every open cover of S contains a finite subcover.

When demonstrating that a set is compact, we must show that every open cover contains a finite subcover. This is difficult to do in general; however, the Heinie-Borel theorem states that a subset of real numbers is compact if and only if it is closed and bounded. To prove the Heini-Borel theorem, we first reequire the following lemma.

Lemma 1.1. *Let $S \subseteq \mathbb{R}$ be non-empty, closed, and bounded. Then, S has a maximum and minimum element.*

Proof. We will prove that the least upper bound is an element of S. A similar proof that the greatest lower bound is an element of S is left as an exercise.

Since S is bounded above, $m = \sup S \in \mathbb{R}$. Let $\epsilon \in \mathbb{R}_{>0}$. Since m is a least upper bound of S, $m - \epsilon$ is not an upper bound of S. If $m \notin S$, then there exists an $x \in S$ such that $m - \epsilon < x < m$, that is, $N^*(m;\epsilon) \cap S \neq \emptyset$. However, this implies that m is an accumulation point of S, which contradicts S being closed since closed sets contain all accumulation points. $\square$

We are now ready to prove the Heine-Borel theorem.

Theorem 1.2. *Let $S \subseteq \mathbb{R}$. Then, S is compact if and only if S is closed and bounded.*

Proof. Suppose that S is compact. For each $n \in \mathbb{N}$, define $I_n = (-n, n)$. Then, each I_n is open and $S \subseteq \bigcup_{n=1}^{\infty} I_n$. Since S is compact, there exists finitely many integers $n_1 < n_2 < \cdots < n_k$ such that

$$S \subseteq \bigcup_{i=1}^{k} I_{n_i} = I_{n_k},$$

It follows that S is bounded since $|x| < n_k$ for all $x \in S$. For the sake of contradiction, suppose that S is not closed. Then, S does not contain all of its accumulation points; let $p \in S' \setminus S$. For each $n \in \mathbb{N}$, define $U_n = \mathbb{R} \setminus [p - 1/n, p + 1/n]$. Since $[p - 1/n, p + 1/n]$ is closed, each U_n is an open set. Moreover,

$$\bigcup_{n=1}^{\infty} U_n = \mathbb{R} \setminus \{p\} \supseteq S.$$

Since S is compact, there exists finitely many integers $n_1 < n_2 < \cdots < n_k$ such that $S \subseteq \bigcup_{i=1}^{k} U_{n_i}$. In fact, since $U_m \subseteq U_n$ if $m \leq n$, it follows that $S \subseteq U_{n_k}$. However, this implies that $S \cap N(p; 1/n_k) = \emptyset$, which contradicts p being an accumulation point of S.

Conversely, suppose that S is closed and bounded. Let $\mathcal{F}$ denote an open cover of S. For each $x \in \mathbb{R}$ define $S_x = S \cap (-\infty, x]$ and let B denote the set of x such that S_x is covered by a finite number of subsets from $\mathcal{F}$. If S is empty, then it is clearly compact; hence, we assume that S is non-empty. Therefore, Lemma 1.1 states that S has a minimum, say d. Also, $S_d = \{d\}$, which is covered by a single subset from $\mathcal{F}$; so, B is non-empty. For the sake of contradiction, suppose that B is bounded above and let $m = \sup B$. If $m \in S$, there exists an $F_0 \in \mathcal{F}$ such that $m \in F_0$. Since F_0 is open, there exists an interval $[a, b] \subsetneq F_0$ such that $a < m < b$. Since $a < m$ and $m = \sup B$, it follows that there exists an $x \in B$ such that $a < x < m$. Therefore, there exists $F_1, \ldots, F_k \in \mathcal{F}$ that cover S_a. However, this implies that $F_0, F_1, \ldots, F_k$ cover S_b, which contradicts $m = \sup B$. If $m \notin S$, then m is not a boundary point of S since S is closed. Therefore, there exists an $\epsilon \in \mathbb{R}_{>0}$ such that $N(m; \epsilon) \cap S = \emptyset$. However, this implies that $S_{m-\epsilon} = S_{m+\epsilon/2}$. Since $m - \epsilon \in B$, it follows that $S_{m+\epsilon/2}$ is in B, which contradicts $m = \sup B$. Therefore, B is not bounded above. Since S is bounded above, there exists a $p \in B$ such that $p > \sup S$. Furthermore, $S_p = S$ and since $p \in B$ we conclude that S is compact. $\square$