Monotone and Cauchy Sequences

Thomas R. Cameron

October 3, 2025

1 Cauchy Sequences

Let $s\colon \mathbb{N} \to \mathbb{R}$ be a sequence. We say that s is a *Cauchy sequence* if for all $\epsilon \in \mathbb{R}_{>0}$ there is a $N \in \mathbb{N}$ such that $|s_n - s_m| < \epsilon$ whenever $n, m \geq N$. Over complete ordered fields, Cauchy sequences are exactly those sequences that are convergent. We will establish this result for the complete ordered field $\mathbb{R}$; the first half of this result is shown in the following proposition.

Proposition 1.1. *Every convergent sequence is a Cauchy sequence.*

Proof. Suppose that $s\colon \mathbb{N} \to \mathbb{R}$ converges to $L \in \mathbb{R}$ and let $\epsilon \in \mathbb{R}_{>0}$. Then, there exists a $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow |s_n - L| < \frac{\epsilon}{2}.$$

Now, for $n, m \geq N$ we have

$$\begin{aligned} |s_n - s_m| &= |s_n - L + L - s_m| \\ &\leq |s_n - L| + |s_m - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

The following lemma shows that all Cauchy sequences are bounded. We have established this result for rational Cauchy sequences; note, that the same argument applies to real Cauchy sequences.

Lemma 1.2. *Every Cauchy sequence is bounded.*

The following result is known as the Bolzano-Weierstrass theorem.

Theorem 1.3 (Bolzano-Weierstrass Theorem)**.** *Let $S \subseteq \mathbb{R}$. If S is bounded and contains infinitely many points, then S' is non-empty.*

Using Lemma 1.2 and Theorem 1.3, we can prove that every Cauchy sequence is convergent.

Theorem 1.4. *Every Cauchy sequence is convergent.*

Proof. Let $s\colon \mathbb{N} \to \mathbb{R}$ be a Cauchy sequence. Also, let $\operatorname{rng}(s) = \{s_n : n \in \mathbb{N}\}$ denote the range of s. Suppose that $\operatorname{rng}(s)$ is finite. Let $\epsilon \in \mathbb{R}_{>0}$ be the minimum distance between distinct elements of $\operatorname{rng}(s)$. Since s is cauchy, there is a $N \in \mathbb{N}$ such that $|s_n - s_m| < \epsilon$. Therefore, $s_n = s_N$ for all $n \geq N$, so $\lim_{n\to\infty} s_n = s_N$.

Suppose that $\operatorname{rng}(s)$ is infinite. Then, the Bolzano-Weierstrass theorem states that $\operatorname{rng}(s)$ has an accumulation point, say $L \in \mathbb{R}$. Let $\epsilon \in \mathbb{R}_{>0}$. Since s is Cauchy, there is a $N \in \mathbb{N}$ such that

$$n, m \geq N \Rightarrow |s_n - s_m| < \frac{\epsilon}{2}.$$

Since L is an accumulation point, $N(L;\epsilon/2) \cap \operatorname{rng}(s)$ contains infinitely many points. Thus, there exists a $m \geq N$ such that $s_m \in N(L;\epsilon/2)$. Therefore, $n \geq N$ implies that

$$\begin{aligned} |s_n - L| &= |s_n - s_m + s_m - L| \\ &\leq |s_n - s_m| + |s_m - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus, $\lim_{n\to\infty} s_n = L$. $\square$