Cardinality

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September 5, 2025

1 Countable Sets

Two sets A and B have the same cardinality, or are *equinumerous*, if there is a bijection $f\colon A \to B$. We denote equinumerous sets by $A \sim B$. For each $n \in \mathbb{N}$, define $I_n = \{1, 2, \ldots, n\}$. A set A is *finite* if $A = \emptyset$ or $A \sim I_n$ for some $n \in \mathbb{N}$. If A is not finite, then it is *infinite*. Moreover, A is *denumerable* if $A \sim \mathbb{N}$. If A is finite or denumerable, then A is *countable*; otherwise, A is *uncountable*.

In this section, we prove that the integers and rational numbers are countable sets. To begin, consider the following preliminary results.

Lemma 1.1. *Let B be a countable set and let $A \subseteq B$. Then, A is countable.*

Proof. If A is finite, then we are done. Suppose that A is infinite, then B must also be infinite. Since B is denumerable, there exists a bijection $f\colon \mathbb{N} \to B$. Now, define

$$S = \{n \in \mathbb{N}\colon f(n) \in A\}$$

Since A is infinite and f is surjective, S must be infinite. The Well-Ordering property of $\mathbb{N}$ implies that S has a least element, say s_1. Similarly, the set $S \setminus \{s_1\}$ has a least element, say s_2. In general, having chosen $s_1, \ldots, s_k$, let s_{k+1} be the least element of $S \setminus \{s_1, \ldots, s_k\}$.

Now, define $g\colon \mathbb{N} \to \mathbb{N}$ by $g(k) = s_k$. Since S is infinite, g is defined for every $k \in \mathbb{N}$. Also, since $a_{k+1} \notin \{a_1, \ldots, a_k\}$, g is injective. Thus, the composition $f \circ g$ is an injection from $\mathbb{N}$ into A. In fact, this composition is a surjection. Indeed, let $a \in A$. Then, since f is surjective, there exists a $n \in \mathbb{N}$ such that $f(n) = a$. Thus, $n \in S$ and there exists a $k \in \mathbb{N}$ such that $g(k) = n$. Therefore, $f(g(k)) = a$. □

Theorem 1.2. *Let A be a non-empty set. Then, the following conditions are equivalent.*

(a) A is countable.

(b) There exists an injection $f\colon A \to \mathbb{N}$.

(c) There exists a surjection $f\colon \mathbb{N} \to A$.

Next, we show that the Cartesian product of two countable sets is also countable.

Corollary 1.3. *Let A and B be countable sets. Then, $A \times B$ is countable.*

Proof. Since A and B are countable, Theorem 1.2 implies that there are injections $f\colon A \to \mathbb{N}$ and $g\colon B \to \mathbb{N}$. Define $h\colon A \times B \to \mathbb{N}$ by

$$h(a, b) = 2^{f(a)} \cdot 3^{g(b)},$$

for all $(a, b) \in A \times B$. Suppose that $h(a, b) = h(a', b')$. Since the prime factorization of a number is unique, up to the order of the factors, it follows that $f(a) = f(a')$ and $g(b) = g(b')$. Since f and g are injective, it follows that $a = a'$ and $b = b'$. Therefore, h is injective, and Theorem 1.2 impliles that $A \times B$ is countable. □

Since the integers correspond to the equivalence classes of an equivalence relation on $\mathbb{N} \times \mathbb{N}$, Corollary 1.3 implies that the integers are countable. Moreover, since the ratoinals correspond to the equivalence classes of an equivalence relation on $\mathbb{Z} \times \mathbb{Z}$, Corollary 1.3 implies that the rationals are countable. We summarize these results in the corollary below.

Corollary 1.4. *The integers $\mathbb{Z}$ and the rationals $\mathbb{Q}$ are both countable.*

2 Uncountable Sets

In this section, we show that the set of real numbers is uncountable.

Theorem 2.1. *The set of real numbers is uncountable.*

Proof. By Lemma 1.1, the subset if any countable set is also countable. Hence, it suffices to show that the interval $J = (0, 1)$ is uncountable. For the sake of contradiction, suppose that J is countable. Then, we could list its elements as follows

$$J = \{x_n \colon n \in \mathbb{N}\}.$$

Each element of J can be written as an infinite decimal expansions

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}\cdots, \\ x_2 &= 0.a_{21}a_{22}a_{23}\cdots, \\ x_3 &= 0.a_{31}a_{32}a_{33}\cdots, \\ &\vdots \end{aligned}$$

where each $a_{ij} \in \{0, 1, \ldots, 9\}$. We now define the real number $y = 0.b_1b_2b_3\cdots$ by

$$b_i = \begin{cases} 2 & \text{if } a_{ii} \neq 2, \\ 3 & \text{if } a_{ii} = 2. \end{cases}$$

Since $y \neq x_i$ for any $i \in \mathbb{N}$, we have a contradiction to our assumption that J is countable. □