Calculus with Analytic Geometry II

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1 Taylor Polynomials

We begin with an example where we approximate the value of e using the Taylor polynomial of $f(x) = e^x$ at $x_0 = 0$. In the table below, we report the nth degree Taylor polynomial, $p_n(1)$, and the error bound on $e - p_n(1)$. Recall that the error bound is given by

$$M\frac{|x - x_0|^{n+1}}{(n+1)!},$$

where M is an upperbound on $\left|f^{(n+1)}(t)\right|$ over the interval $[x_0, x]$. In this case, $M = e$.

n	$p_n(x)$	$p_n(1)$	Error Bound
1	$1+x$	2	$\frac{e}{2!} \approx 1.359$
2	$1+x+\frac{x^2}{2}$	2.5	$\frac{e}{3!} \approx 0.453$
3	$1+x+\frac{x^2}{2}+\frac{x^3}{6}$	2.667	$\frac{e}{4!} \approx 0.113$
4	$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}$	2.708	$\frac{e}{5!} \approx 0.0226$
5	$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}$	2.7167	$\frac{e}{6!} \approx 0.0037$
6	$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}$	2.71805	$\frac{e}{7!} \approx 0.00054$
7	$1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+\frac{x^6}{720}+\frac{x^7}{5040}$	2.71825	$\frac{e}{8!} \approx 0.0000674$

Since e is an irrational number (has an infinite decimal expansion) we must use a Taylor polynomial of infinite degree to obtain all digits of e.

2 Taylor Series

Suppose that $f(x)$ is infinitely differentiable at x_0. Then, the Taylor series of $f(x)$ at x_0 is defined as follows

$$\sum_{k=0}^{\infty} f^{(k)}(x_0)\frac{(x-x_0)^k}{k!}.$$

For example, the Taylor series of $f(x) = e^x$ at $x_0 = 0$ is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

As another example, the Taylor series of $f(x) = \ln(x)$ at $x_0 = 1$ is

$$\ln(1) + \frac{1}{1}(x-1) - \frac{1}{1^2}\frac{(x-1)^2}{2!} + \frac{2}{1^3}\frac{(x-1)^3}{3!} - \frac{6}{1^4}\frac{(x-1)^4}{4!} + \cdots = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(x-1)^k}{k}$$

If we plug in a value of x, then the resulting series of numbers either converges or diverges. For example, if we plug $x = 2$ into the Taylor series of $f(x) = e^x$ at $x_0 = 0$, then we obtain

$$\sum_{k=0}^{\infty} \frac{2^k}{k!}.$$

If we plug $x = 2$ into the Taylor series of $f(x) = \ln(x)$ at $x_0 = 1$, then we obtain

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}.$$

3 Exercises

1. Does the Taylor series of $f(x) = e^x$ at $x_0 = 0$ converge when $x = 2$? What are all the x values for which this Taylor series converges?

2. Does the Taylor series of $f(x) = \ln(x)$ at $x_0 = 1$ convergen when $x = 2$? What are all the x values for which this Taylor series converges?