Series

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September 25, 2025

1 Series

A sequence of partial sums is a sequence where the nth term is defined as the sum of n numbers. For instance,

$$\{s_n\}_{n=1}^{\infty} = \left\{ s_n = \sum_{k=1}^{n} u_k \right\}_{n=1}^{\infty}$$

An infinite series, or series for short, is the limit of a sequence of partial sums, that is,

$$\sum_{k=1}^{\infty} u_k = \lim_{n\to\infty} s_n$$

The series is said to converge if the sequence of partial sums converges; otherwise, the series diverges.

For example, consider the series $\sum_{k=1}^{\infty} (-1)^k$, which corresponds to the following sequence of partial sums:

$$\begin{aligned} s_1 &= -1 \\ s_2 &= 0 \\ s_3 &= -1 \\ s_4 &= 0 \\ &\vdots \end{aligned}$$

This sequence of partial sums diverges; hence, the series diverges.

As another example, consider the series $\sum_{k=0}^{\infty} \frac{1}{2^k}$, which has sequence of partial sums

$$\begin{aligned} s_0 &= 1 \\ s_1 &= 1 + \frac{1}{2} = \frac{3}{2} \\ s_2 &= \frac{3}{2} + \frac{1}{4} = \frac{7}{4} \\ s_3 &= \frac{7}{4} + \frac{1}{8} = \frac{15}{8} \\ &\vdots \\ s_n &= \frac{2^{n+1}-1}{2^n}, \end{aligned}$$

which converges to 2.

As another example, consider the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$, which has sequence of partial sums

$$\begin{aligned}
s_1 &= \frac{1}{2} \\
s_2 &= \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3} \\
s_3 &= \frac{2}{3} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4} \\
s_4 &= \frac{3}{4} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5} \\
&\vdots \\
s_n &= \frac{n}{n+1},
\end{aligned}$$

which converges to 1.