Sequences

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1 Sequences

Each time we have solved a geometric problem (area, volume, surface area), we have first approximated the solution as a sum. Then, we applied a limit to transform our sum into a Riemann sum, which we recognize as a definite integral. The mathematical concept underlying this methedology is known as a sequence. Today, we introduce the sequence in its general form. This investigation will lead us to the representation of functions via series, which is a powerful tool in the study of differential equations.

A sequence is a unending succession of numbers, called terms. It is understood that the terms have a definite order, that is, there is a first term a_1, a second term a_2, a third term a_3, and so on. Such a sequence can be written as

$$a_1,\ a_2,\ a_3,\ \ldots.$$

Due to this definite order, sequences can be viewed as functions from the positive integers to the real numbers. For example, the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$$

can be written as the function $f(n) = \frac{n}{n+1}$. As another example, the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

can be written as the function $f(n) = \frac{1}{2^n}$. The function of a sequence can also be written using bracket notation: $\{f(n)\}_{n=1}^{\infty}$. For instance, the sequences above can be written as $\{\frac{n}{n+1}\}_{n=1}^{\infty}$ and $\{\frac{1}{2^n}\}_{n=1}^{\infty}$, respectively.

A sequence is said to converge if the terms of the sequence get closer to a single value If the terms of a sequence do not get closer to a single value, then we say that the sequence diverges. Examples of convergent and divergent sequences are shown in Figure 1.

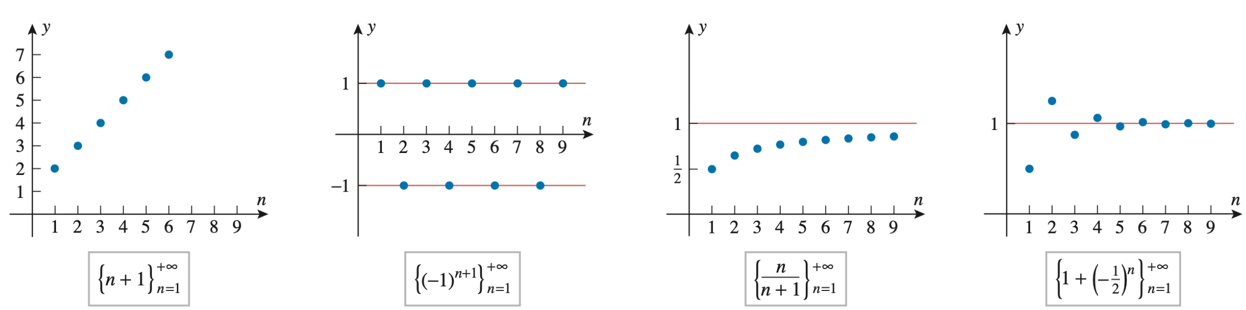

Figure 1: Examples of convergent and divergent sequences

More formally, we say that the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the value L if for any $\epsilon > 0$ there is a positive integer N such that $|a_n - L| < \epsilon$ for all $n > N$. In this case, we write $\lim_{n\to\infty} a_n = L$. An illustration of this definition is given in Figure 2

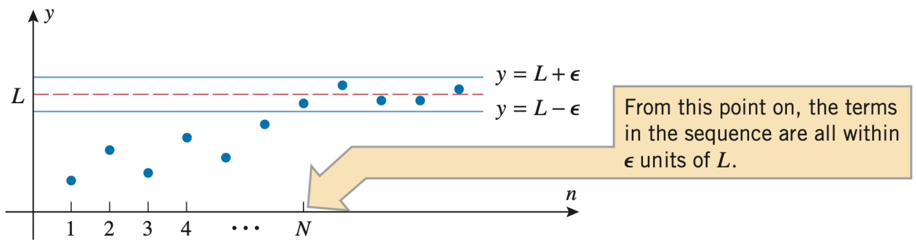

Figure 2: Illustration of formal definition of a convergent sequence

As an example of the formal definition, we will show that the sequence $\{\frac{n}{n+1}\}_{n=1}^{\infty}$ converges to 1. For each $\epsilon > 0$, Let $N = \frac{1-\epsilon}{\epsilon}$. Then, $n > N$ implies that $n > \frac{1-\epsilon}{\epsilon}$, i.e., $n+1 > \frac{1}{\epsilon}$. Therefore,

$$\begin{aligned}\left|\frac{n}{n+1} - 1\right| &= \left|\frac{n-(n+1)}{n+1}\right| \\ &= \frac{1}{n+1} < \epsilon.\end{aligned}$$

If a sequence does not converge to a finite number L, then it is said to diverge. A sequence that diverges satisfies the following: For all finite numbers L, there exists an $\epsilon > 0$ such that for all positive integers N, there is a $n > N$ such that $|a_n - L| \geq \epsilon$. For example, the sequence $\{n\}_{n=1}^{\infty}$ diverges. Let L be any finite number and define $\epsilon = 1$. Then for any positive integer N, there is an $n > N$ such that $|n - L| \geq \epsilon$.