Power Series Solutions to Differential Equations

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1 Differential Equations

A differential equation relates an unknown function to its derivatives. The order of the differential equation is the highest derivative that occurs. For example,

$$y' - y = 0 \tag{1}$$

is a first order differential equation. As another example,

$$y'' + y = 0, \tag{2}$$

is a second order differential equation.

2 General Solutions

Given a differential equation, the goal is to find a family of functions $y(x)$ that satisfy the equation. For example, $y = ae^x$ satisfies the equation in (1), for any constant a. Hence, we say that $y = ae^x$ forms a general solution for the differential equation in (1).

Similarly, $y_1 = a\cos(x)$ and $y_2 = b\sin(x)$ both satisfy the equation in (2). Moreover, y_1 and y_2 are not constant multiples of each other. For this reason, we say that

$$y = a\cos(x) + b\sin(x)$$

forms a general solution for the differential equation in (2).

Power series provide a method for determining the general solution of a differential equation. For example, consider the differential equation in (1) and assume that the solution can be written as a power series

$$y = \sum_{k=0}^{\infty} c_k x^k,$$

with a positive radius of convergence. Then, all derivatives of y exist and can be attained with term-by-term differentiation. In particular,

$$y' = \sum_{k=0}^{\infty} kc_k x^{k-1} = \sum_{n=1}^{\infty} kc_k x^{k-1} = \sum_{k=0}^{\infty} (k+1)c_{k+1} x^k.$$

Plugging the power series of y and y' into the differential equation gives

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} (k+1)c_{k+1} x^k - \sum_{k=0}^{\infty} c_k x^k \\ &= \sum_{k=0}^{\infty} \left[(k+1)c_{k+1} - c_k\right] x^k. \end{aligned}$$

In order for the series to equal zero, it follows that $(k+1)c_{k+1} = c_k$, for all $k \geq 0$. Let c_0 be an arbitrary constant, then

$$
\begin{aligned}
c_1 &= \frac{c_0}{1} \\
c_2 &= \frac{c_1}{2} = \frac{c_0}{2 \cdot 1} \\
&\vdots \\
c_k &= \frac{c_0}{k!}
\end{aligned}
$$

So, the general solution is of the form

$$y(x) = c_0 \sum_{k=0}^{\infty} \frac{1}{k!} x^n = c_0 e^x.$$

Consider the differential equation in (2) and assume that the solution can be written as a power series

$$y = \sum_{n=0}^{\infty} a_n x^n,$$

with a positive radius of convergence. Then, all derivatives of y exist and can be attained with term-by-term differentiation. In particular,

$$y'' = \sum_{k=0}^{\infty} k(k-1)c_k x^{k-2} = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k.$$

Plugging the power series of y and y'' into the differential equation gives

$$
\begin{aligned}
0 &= \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2} x^k + \sum_{k=0}^{\infty} c_k x^k \\
&= \sum_{k=0}^{\infty} \left[(k+2)(k+1)c_{k+2} + c_k\right] x^k.
\end{aligned}
$$

In order for the series to equal zero, it follows that $(k+2)(k+1)c_{k+2} = -c_k$, for all $k \geq 0$. Let c_0 and c_1 be arbitrary constants, then

$$
\begin{aligned}
c_2 &= -\frac{c_0}{2 \cdot 1} & c_3 &= -\frac{c_1}{3 \cdot 2} \\
c_4 &= \frac{c_0}{4!} & c_5 &= \frac{c_1}{5!} \\
&\vdots \\
c_{2k} &= (-1)^k \frac{c_0}{(2k)!} & c_{2k+1} &= (-1)^k \frac{c_1}{(2k+1)!}
\end{aligned}
$$

So, the general solution is of the form

$$y(x) = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = c_0 \cos(x) + c_1 \sin(x).$$

Consider the differential equation

$$y'' - xy = 0,$$

and assume that the solution can be written as a power series

$$y = \sum_{k=0}^{\infty} c_k x^k,$$

with a positive radius of convergence. Then, all derivatives of y exist and can be attained with term-by-term differentiation. In particular,

$$y'' = \sum_{k=0}^{\infty} k(k-1)c_k x^{k-2} = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k.$$

Also, note that

$$xy = \sum_{k=0}^{\infty} c_k x^{k+1} = \sum_{k=1}^{\infty} c_{k-1}x^k$$

Plugging the power series of xy and y'' into the differential equation gives

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} c_{k-1}x^k \\ &= 2c_2 + \sum_{k=1}^{\infty}\left[(k+2)(k+1)c_{k+2} - c_{k-1}\right]x^k \end{aligned}$$

In order for the series to equal zero, it follows that $c_2 = 0$ and $(k+2)(k+1)c_{k+2} = c_{k-1}$, for all $k \geq 1$. Therefore,

$$\begin{aligned} c_{3k} &= \frac{c_0}{(2\cdot 3)\cdot(5\cdot 6)\cdots((3k-1)\cdot(3k))} \\ c_{3k+1} &= \frac{c_1}{(3\cdot 4)\cdot(6\cdot 7)\cdots((3k)\cdot(3k+1))} \\ c_{3k+2} &= 0, \end{aligned}$$

for $k \geq 1$, where c_0 and c_1 are arbitrary constants. So, the general solution is of the form

$$y = c_0\left(1 + \sum_{k=1}^{\infty}\frac{x^{3k}}{(2\cdot 3)\cdots(3k-1)\cdot(3k)}\right) + c_1\left(x + \sum_{k=1}^{\infty}\frac{x^{3k+1}}{(3\cdot 4)\cdots(3k)\cdot(3k+1)}\right)$$

Consider the differential equation

$$x^2y'' + xy' + x^2y = 0,$$

and assume that the solution can be written as a power series

$$y = \sum_{k=0}^{\infty} c_k x^k,$$

with a positive radius of convergence. Then, all derivatrives of y exist and can be attained with term-by-term differentiation. In particular,

$$y' = \sum_{k=0}^{\infty}(k+1)c_{k+1}x^k \text{ and } y'' = \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k.$$

Also, note that

$$\begin{aligned} x^2y &= \sum_{k=0}^{\infty} c_k x^{k+2}, \\ xy' &= \sum_{k=0}^{\infty}(k+1)c_{k+1}x^{k+1}, \\ x^2y'' &= \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^{k+2}. \end{aligned}$$

We can write the series for xy' as follows

$$xy' = c_1 x + \sum_{k=1}^{\infty}(k+1)c_{k+1}x^{k+1} = c_1 x + \sum_{k=0}^{\infty}(k+2)c_{k+2}x^{k+2}.$$

Plugging the power series of x^2y, xy', and x^2y'' into the differential equation gives

$$\begin{aligned} 0 &= \sum_{k=0}^{\infty} c_k x^{k+2} + c_1 x + \sum_{k=0}^{\infty}(k+2)c_{k+2}x^{k+2} + \sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^{k+2} \\ &= c_1 x + \sum_{k=0}^{\infty} x^{k+2}\left((k+2)^2 c_{k+2} + c_k\right). \end{aligned}$$

In order for the series to equal zero, it follows that $c_1 = 0$ and $(k+2)^2 c_{k+2} = -c_k$, for all $k \geq 0$. Therefore,

$$c_{2k+1} = 0 \text{ and } c_{2k} = \frac{(-1)^k c_0}{2^{2k}(k!)^2},$$

for all $k \geq 0$, where c_0 is an arbitrary constant. It is important to note that in this example we are only able to find one independent solution.