# Differentiation of Power Series

Thomas R. Cameron

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## 1 Differentiation of Power Series

Suppose that $\sum_{k=0}^{\infty} c_k(x-x_0)^k$ has a positive radius of convergence $R > 0$. Then, for each $x \in (x_0-R, x_0+R)$ define $f(x) = \sum_{k=0}^{\infty} c_k(x-x_0)^k$. Note that the function $f$ is defined by the value of the absolutely convergent series. Moreover, for each $x \in (x_0-R, x_0+R)$ there is a $h > 0$ small enough so that $x+h \in (x_0-R, x_0+R)$. Therefore, $f(x+h) = \sum_{k=0}^{\infty} c_k(x+h-x_0)^k$. Recall the limit definition of the derivative,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

In the context of the power series, we have

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left(\sum_{k=0}^{\infty} c_k(x+h-x_0)^k - \sum_{k=0}^{\infty} c_k(x-x_0)^k\right) \\ &= \lim_{h\to 0} \sum_{k=0}^{\infty} c_k\left(\frac{(x+h-x_0)^k-(x-x_0)^k}{h}\right) \\ &= \sum_{k=0}^{\infty} c_k\left(\lim_{h\to 0}\frac{(x+h-x_0)^k-(x-x_0)^k}{h}\right). \end{aligned}$$

Note that the last equality only holds for absolutely convergent series. Furthermore,

$$\lim_{h\to 0}\frac{(x+h-x_0)^k-(x-x_0)^k}{h} = \begin{cases} 0 & \text{if } k=0, \\ k(x-x_0)^{k-1} & \text{if } k\geq 1. \end{cases}$$

Therefore,

$$f'(x) = \sum_{k=1}^{\infty} c_k k(x-x_0)^{k-1} = \sum_{k=0}^{\infty} c_{k+1}(k+1)(x-x_0)^k.$$

So, when a function is represented by a power series with a positive radius of convergence, its derivative can be represented by a power series through term by term differentiation. In fact, the power series for the derivative has the same radius of convergence as the original. Indeed,

$$\begin{aligned} \lim_{k\to\infty}\frac{|c_{k+2}(k+2)(x-x_0)^{k+1}|}{|c_{k+1}(k+1)(x-x_0)^k|} &= \lim_{k\to\infty}\frac{|c_{k+2}|}{|c_{k+1}|}|x-x_0| \\ &= \lim_{k\to\infty}\frac{|c_{k+1}|}{|c_k|}|x-x_0| \\ &= \lim_{k\to\infty}\frac{|c_{k+1}(x-x_0)^{k+1}|}{|c_k(x-x_0)^k|}, \end{aligned}$$

so the limit of the ratios is the same for the series representing $f(x)$ and $f'(x)$.

We summarize these results in the following theorem.

**Theorem 1.1.** *Let* $f(x) = \sum_{k=0}^{\infty} c_k (x-x_0)^k$ *have a positive radius of convergence* $R > 0$*. Then,*

$$f'(x) = \sum_{k=1}^{\infty} c_k k (x-x_0)^{k-1} = \sum_{k=0}^{\infty} c_{k+1}(k+1)(x-x_0)^k$$

*has the same radius of convergence* $R$*.*

As an example, consider the power series representation of the natural exponential function

$$f(x) = e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

Then, we can represent the derivative as follows

$$\begin{aligned} f'(x) &= \sum_{k=1}^{\infty} \frac{k}{k!} x^{k-1} \\ &= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} x^k. \end{aligned}$$

Hence, $f'(x) = f(x)$, as expected for the natural exponential function. Note that the radius of convergence for both $f(x)$ and $f'(x)$ is infinity.

As another example, consider the power series representation of the natural log function

$$f(x) = \ln(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k,$$

which has a radius of convergence $R = 1$ and interval of convergence $(0, 2]$. We can represent the derivative as follows

$$\begin{aligned} f'(x) &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} k (x-1)^{k-1} \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} (x-1)^{k-1} \\ &= \sum_{k=0}^{\infty} (-1)^k (x-1)^k \\ &= \sum_{k=0}^{\infty} (1-x)^k, \end{aligned}$$

which has radius of convergence $R = 1$ and interval of convergence $(0, 2)$.