Calculus in Polar Coordinates

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1 Tangent Lines, Arc Length and Surface Area

Consider the function $r = f(\theta)$, where f is a differentiable function in θ. In cartesian coordinates, we have $x = f(\theta)\cos(\theta)$ and $y = f(\theta)\sin(\theta)$. Therefore,

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$$

For example, if $r = \sin(\theta)$, for $0 \leq \theta \leq \pi$, then

$$\frac{dy}{dx} = \frac{2\sin(\theta)\cos(\theta)}{\cos^2(\theta) - \sin^2(\theta)}$$

Hence, the graph of $r = \sin(\theta)$ has a horizontal tangent line when $\theta = 0$ and $\theta = \pi$, and a vertical tangent line when $\theta = \pi/4$ and $\theta = 3\pi/4$.

Let $x = f(\theta)\cos(\theta)$ and $y = f(\theta)\sin(\theta)$, for $\alpha \leq \theta \leq \beta$. Suppose that the curve parameterized by x and y is never traced more than once. Then, the arc length over $[\alpha, \beta]$ is given by

$$L = \int_\alpha^\beta \sqrt{x'(\theta)^2 + y'(\theta)^2}d\theta = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}d\theta.$$

For example, if $r = \sin(\theta)$, $\alpha = 0$, $\beta = \pi$, then

$$\begin{aligned} L &= \int_0^\pi \sqrt{\sin^2(\theta) + \cos^2(\theta)}d\theta \\ &= \int_0^\pi d\theta = \pi. \end{aligned}$$

Next, suppose that the curve parameterized by x and y is never below the x-axis. Furthermore, suppose the curve is revolved about the x-axis. Then, the area of the surface of revolution is given by

$$S = \int_\alpha^\beta 2\pi f(\theta)\sin(\theta)\sqrt{f(\theta)^2 + f'(\theta)^2}d\theta.$$

For example, if $r = \sin(\theta)$, $\alpha = \pi/4$, and $\beta = 3\pi/4$, then

$$\begin{aligned} S &= \int_{\pi/4}^{3\pi/4} 2\pi\sin^2(\theta)\sqrt{\sin^2(\theta) + \cos^2(\theta)}d\theta \\ &= 2\pi\int_{\pi/4}^{3\pi/4} \sin^2(\theta)d\theta \\ &= 2\pi\left(-\frac{1}{2}\cos(\theta)\sin(\theta)\Big|_{\pi/4}^{3\pi/4} + \frac{1}{2}\theta\Big|_{\pi/4}^{3\pi/4}\right) \\ &= 2\pi\left(-\frac{1}{2}\left(-\frac{1}{2} - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)\right) \\ &= 2\pi\left(\frac{1}{2} + \frac{\pi}{4}\right) = \pi\left(1 + \frac{\pi}{2}\right). \end{aligned}$$

2 Area in Polar Coordinates

Consider the function $r = f(\theta)$, where f is continuous and non-negative over the interval $\alpha \leq \theta \leq \beta$. The region bounded by the graph of f and the radial lines $\theta = \alpha$ and $\theta = \beta$ is shown in Figure 1 (left).

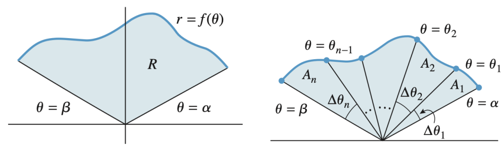

Figure 1: Region bounded by $r = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$.

To approximate the area of this region, we subdivide the interval $[\alpha, \beta]$ into n subintervals

$$\alpha = \theta_0 < \theta_1 < \cdots < \theta_{n-1} < \theta_n = \beta.$$

These n subintervals correspond to n sectors of the region as shown in Figure 1 (right). The area of the kth sector can be approximated by a sector of a circle of radius $f(\theta_{k^*})$ and central angle $\Delta\theta_k$, as shown in Figure 2, which is given by $\frac{1}{2}\Delta\theta_k f(\theta_{k^*})^2$.

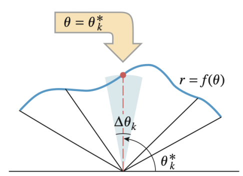

Figure 2: Region bounded by $r = f(\theta)$, $\theta = \alpha$, and $\theta = \beta$.

Therefore, the area of the entire region can be approximated by the following summation

$$A \approx \sum_{i=1}^{n} \frac{1}{2}\Delta\theta_i f(\theta_{i^*})^2.$$

Taking the limit as $n \to \infty$ gives us

$$\begin{aligned} A &= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}\Delta\theta_i f(\theta_{i^*})^2 \\ &= \int_{\alpha}^{\beta} \frac{1}{2} f(\theta)^2 d\theta, \end{aligned}$$

where $0 < \beta - \alpha \leq 2\pi$.

For example, consider $r = \cos(2\theta)$. The pedal determined by the interval $[-\pi/4, \pi/4]$ has area

$$\begin{aligned} A &= \int_{-\pi/4}^{\pi/4} \frac{1}{2}\cos^2(2\theta) d\theta \\ &= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \cos^2(u) du \\ &= \frac{1}{4}\left(\frac{1}{2}\cos(u)\sin(u) + \frac{1}{2}u\right)\Bigg|_{-\pi/2}^{\pi/2} = \frac{\pi}{8}. \end{aligned}$$