Math 140 Worksheet Week 13 Solutions

Week 13: Riemann Sums and Exact Area

Throughout, we divide the interval $[a, b]$ into n subintervals of equal width

$$\Delta x = \frac{b-a}{n},$$

and define

$$x_i = a + i\Delta x, \qquad i = 0, 1, 2, \ldots, n.$$

Thus,

$$x_0 = a, \qquad x_n = b.$$

1. (Linear Function)

Evaluate

$$\int_a^b x\,dx$$

using the sample points

$$c_i = \frac{x_{i-1} + x_i}{2}.$$

Since $f(x) = x$, we have

$$f(c_i)\Delta x = c_i \Delta x = \frac{x_{i-1} + x_i}{2}\Delta x.$$

Because

$$x_i - x_{i-1} = \Delta x,$$

we may write

$$\frac{x_{i-1} + x_i}{2}\Delta x = \frac{x_i^2 - x_{i-1}^2}{2}.$$

Therefore,

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} \frac{x_i^2 - x_{i-1}^2}{2}.$$

This is a telescoping sum:

$$\sum_{i=1}^{n} \frac{x_i^2 - x_{i-1}^2}{2} = \frac{x_n^2 - x_0^2}{2} = \frac{b^2 - a^2}{2}.$$

Hence,

$$\int_a^b x\,dx = \frac{b^2 - a^2}{2}.$$

2. (Exponential Function)
Evaluate

$$\int_a^b e^x \, dx$$

using the sample points

$$c_i = x_{i-1} + \ln\left(\frac{e^{\Delta x} - 1}{\Delta x}\right).$$

Since $f(x) = e^x$, we compute

$$\begin{aligned} f(c_i)\Delta x &= e^{c_i}\Delta x \\ &= e^{x_{i-1}+\ln\left(\frac{e^{\Delta x}-1}{\Delta x}\right)}\Delta x \\ &= e^{x_{i-1}}\left(\frac{e^{\Delta x} - 1}{\Delta x}\right)\Delta x \\ &= e^{x_{i-1}}(e^{\Delta x} - 1). \end{aligned}$$

Since

$$x_i = x_{i-1} + \Delta x,$$

we have

$$e^{x_{i-1}}e^{\Delta x} = e^{x_i},$$

so

$$f(c_i)\Delta x = e^{x_i} - e^{x_{i-1}}.$$

Therefore,

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n}\left(e^{x_i} - e^{x_{i-1}}\right).$$

This telescopes:

$$\sum_{i=1}^{n}\left(e^{x_i} - e^{x_{i-1}}\right) = e^{x_n} - e^{x_0} = e^b - e^a.$$

Hence,

$$\int_a^b e^x \, dx = e^b - e^a.$$

3. (Reciprocal Function)
Assume $0 < a < b$. Evaluate

$$\int_a^b \frac{1}{x} \, dx$$

using the sample points

$$c_i = \frac{x_i - x_{i-1}}{\ln x_i - \ln x_{i-1}}.$$

Since $f(x) = \frac{1}{x}$, we have

$$f(c_i)\Delta x = \frac{1}{c_i}\Delta x.$$

From the definition of c_i,

$$c_i = \frac{x_i - x_{i-1}}{\ln x_i - \ln x_{i-1}},$$

so

$$\frac{1}{c_i} = \frac{\ln x_i - \ln x_{i-1}}{x_i - x_{i-1}}.$$

Multiplying by $\Delta x = x_i - x_{i-1}$ gives

$$f(c_i)\Delta x = \ln x_i - \ln x_{i-1}.$$

Therefore,

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n}(\ln x_i - \ln x_{i-1}).$$

This telescopes:

$$\sum_{i=1}^{n}(\ln x_i - \ln x_{i-1}) = \ln x_n - \ln x_0 = \ln b - \ln a.$$

Hence,

$$\int_a^b \frac{1}{x}\,dx = \ln b - \ln a.$$

4. (Reciprocal Square Function)
Assume $0 < a < b$. Evaluate

$$\int_a^b \frac{1}{x^2}\,dx$$

using the sample points

$$c_i = \sqrt{x_{i-1}x_i}.$$

Since $f(x) = \frac{1}{x^2}$, we have

$$f(c_i)\Delta x = \frac{1}{c_i^2}\Delta x.$$

Because

$$c_i = \sqrt{x_{i-1}x_i},$$

it follows that

$$c_i^2 = x_{i-1}x_i.$$

Thus,

$$f(c_i)\Delta x = \frac{\Delta x}{x_{i-1}x_i}.$$

Now,

$$\Delta x = x_i - x_{i-1},$$

so

$$f(c_i)\Delta x = \frac{x_i - x_{i-1}}{x_{i-1}x_i} = \frac{1}{x_{i-1}} - \frac{1}{x_i}.$$

Therefore,

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n}\left(\frac{1}{x_{i-1}} - \frac{1}{x_i}\right).$$

This telescopes:

$$\sum_{i=1}^{n} \left(\frac{1}{x_{i-1}} - \frac{1}{x_i} \right) = \frac{1}{x_0} - \frac{1}{x_n} = \frac{1}{a} - \frac{1}{b}.$$

Hence,

$$\int_a^b \frac{1}{x^2}\, dx = \frac{1}{a} - \frac{1}{b}.$$

5. (Reciprocal Square Root Function)
Assume $0 < a < b$. Evaluate

$$\int_a^b \frac{1}{\sqrt{x}}\, dx$$

using the sample points

$$c_i = \left(\frac{\sqrt{x_{i-1}} + \sqrt{x_i}}{2} \right)^2.$$

Since $f(x) = \frac{1}{\sqrt{x}}$, we have

$$f(c_i)\Delta x = \frac{1}{\sqrt{c_i}} \Delta x.$$

From the definition of c_i,

$$\sqrt{c_i} = \frac{\sqrt{x_{i-1}} + \sqrt{x_i}}{2}.$$

Therefore,

$$\frac{1}{\sqrt{c_i}} = \frac{2}{\sqrt{x_{i-1}} + \sqrt{x_i}}.$$

Multiplying by $\Delta x = x_i - x_{i-1}$ gives

$$f(c_i)\Delta x = \frac{2(x_i - x_{i-1})}{\sqrt{x_{i-1}} + \sqrt{x_i}}.$$

Now use

$$x_i - x_{i-1} = (\sqrt{x_i} - \sqrt{x_{i-1}})(\sqrt{x_i} + \sqrt{x_{i-1}}),$$

to obtain

$$f(c_i)\Delta x = 2(\sqrt{x_i} - \sqrt{x_{i-1}}).$$

Therefore,

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} 2(\sqrt{x_i} - \sqrt{x_{i-1}}).$$

This telescopes:

$$\sum_{i=1}^{n} 2(\sqrt{x_i} - \sqrt{x_{i-1}}) = 2(\sqrt{x_n} - \sqrt{x_0}) = 2(\sqrt{b} - \sqrt{a}).$$

Hence,

$$\int_a^b \frac{1}{\sqrt{x}}\, dx = 2(\sqrt{b} - \sqrt{a}).$$

6. (Reflection)
In each of the previous problems, the Riemann sum telescopes. Explain why this happens. What role does the function $F(x)$ (an antiderivative of $f(x)$) appear to play in this process?

In each problem, the sample points c_i were chosen so that

$$f(c_i)\Delta x = F(x_i) - F(x_{i-1}),$$

where F is an antiderivative of f. Therefore,

$$\sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big).$$

This is a telescoping sum, so all of the middle terms cancel and we obtain

$$\sum_{i=1}^{n} f(c_i)\Delta x = F(x_n) - F(x_0) = F(b) - F(a).$$

Thus, the antiderivative F is what makes the telescoping structure possible. This pattern leads directly to the Fundamental Theorem of Calculus:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$