Worksheet 5: Derivative Tests and Applications

Math 140: Calculus with Analytic Geometry

Name: _____ **Date:** _____

Directions. Complete this take-home review over several days. Use the Mean Value Theorem, first and second derivative test, and linearization notes as needed. Show sign charts and explain your conclusions; a derivative equal to zero is only a candidate. Give exact values and label important points on sketches. Use additional paper if needed. No graphing software is needed.

1. **An exponential function**

   Let $f(x) = e^{-x^2}$, with domain $\mathbb{R}$.

   (a) Find f' and f'', and factor each derivative. Use a sign chart for f' to find the intervals of increase and decrease and classify all critical numbers using the first derivative test.

   (b) Find the intervals of concavity and all inflection points. Check the classification of each stationary point using the second derivative test.

   (c) Find $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$, and sketch the graph, labeling intercepts, extrema, and inflection points. Does f attain an absolute maximum or minimum on $\mathbb{R}$? Explain. Identify an interval where the graph decreases while concave up, and explain what happens to its tangent slopes there.

2. **A trigonometric function with a nonlinear input**

Let $g(x) = \sin(x^2)$ on $[-\sqrt{\pi}, \sqrt{\pi}]$. Use radians. Only the signs and standard unit-circle values of sine and cosine are needed; no double-angle identities are needed.

(a) Find g' and g''. Find all interior critical numbers. Make a sign chart for g' and use the first derivative test to classify each critical number.

(b) Evaluate g'' at each critical number and use the second derivative test to check your classifications. Explain why evaluating g'' at these points is sufficient for this test, even without finding every interval of concavity.

(c) Compare values at the endpoints and interior critical numbers to find the absolute maximum and minimum on the given interval. Give all inputs where they occur. Sketch the graph, labeling its intercepts and extrema; a complete concavity analysis is not required.

3. **A rational function**

Let $h(x) = \dfrac{x^2}{x^2+3}$, with domain $\mathbb{R}$.

(a) Find and simplify h' and h''. Find all critical numbers, intervals of increase and decrease, and local extrema.

(b) Use the second derivative to find concavity intervals and inflection points. Does an inflection point have to be a critical point of h? Support your answer with this function.

(c) Find $\lim_{x\to\pm\infty} h(x)$ and sketch the graph, labeling its extrema, inflection points, and horizontal asymptote. Does h attain an absolute maximum or minimum on $\mathbb{R}$? Explain.

4. **The Mean Value Theorem and a logarithm bound**

(a) State the Mean Value Theorem, including its hypotheses and conclusion.

(b) Show that $f(x) = \ln x$ on $[1, 2]$ satisfies the hypotheses. Apply the theorem to express $\ln 2$ in terms of $f'(c)$ for some $c \in (1, 2)$.

(c) Use $1 < c < 2$ to prove that $\frac{1}{2} < \ln 2 < 1$. Save this bound for the approximation in Problem 6.

5. **When the second derivative test is inconclusive**

Consider $p(x) = (x-1)^4$, $q(x) = (x-1)^3$, and $r(x) = |x-1|$.

(a) Find the first and second derivative functions of both p and q. Evaluate them at $x = 1$. What conclusion does the second derivative test give for each function?

(b) Use the first derivative test to classify $x = 1$ for each function. Give each extremum as a point on its graph.

(c) Determine whether $(1, 0)$ is an inflection point of p or q. Then find r' for $x < 1$ and $x > 1$ and classify $x = 1$ for r. Explain why the second derivative test is inconclusive for p and q, but does not apply to r.

6. **A logarithm and its tangent approximation**

Let $k(x) = \ln(1 + x^2)$, with domain $\mathbb{R}$.

(a) Find and simplify k' and k''. Determine the intervals of increase and decrease and classify all critical numbers using the first derivative test.

(b) Find the intervals of concavity and all inflection points. Check the extremum classification using the second derivative test.

(c) Find the linearization L at $a = 1$ and use it to estimate $k(1.1) = \ln(2.21)$. Keep $\ln 2$ in your estimate. Is the estimate too high or too low? Justify using concavity between 1 and 1.1. Use the bound from Problem 4 to give lower and upper bounds for the value of your linear estimate.

7. **Position, direction, and turning**

A cart moves along a straight track. Its position, measured in meters from a fixed marker, is

$$s(t) = t^3 - 6t^2 + 9t, \qquad 0 \le t \le 4,$$

where t is measured in seconds. Positive position means the cart is to the right of the marker. Velocity is the rate of change of position; its sign tells the direction of motion. Acceleration is the rate of change of velocity.

(a) Find the velocity $v(t) = s'(t)$ and acceleration $a(t) = s''(t)$. Make a sign chart for velocity. When does the cart move right, when does it move left, and at which interior times does it reverse direction?

(b) Find when the position graph is concave up or down and locate its inflection point. On $(1, 2)$ and $(2, 3)$ the cart moves left. On which interval does its velocity become more negative, and on which does it become less negative? Explain using acceleration.

(c) Compare the positions at the endpoints and interior critical times to find the leftmost and rightmost positions during $[0, 4]$. Give every time at which each is attained. Explain why being momentarily at rest is different from being at the marker.

8. **Building a model for maximum area**

A farmer has 80 m of fencing for a rectangular pen beside a straight river. No fence is needed along the river; all 80 m is used for the other three sides. Let x be the length of each of the two sides perpendicular to the river, and let y be the length of the fenced side parallel to it. Both lengths must be positive. The goal is to choose the dimensions that enclose the greatest area. This is an optimization problem: first express the quantity to be maximized as a function of one variable, then use derivatives to compare possible choices.

(a) Draw and label the pen, marking the unfenced side. Write the equation for the total fencing length. Solve it for y and substitute into $A = xy$ to express area as $A(x)$. Find the allowed interval for x using $x > 0$ and $y > 0$.

(b) Find A' and A''. Find the critical number in the allowed interval. Use derivative signs on each side to show that this choice gives the greatest area, and check with the second derivative test. Give both dimensions and the area, including units.

(c) For an independent check, extend $A(x)$ continuously to the endpoints of the corresponding closed interval. Compare its values at both endpoints and the critical number. What shape do the endpoints represent, and why does the maximizing choice still belong to the original physical domain?