

# Trigonometric Integrals

Math 140: Calculus with Analytic Geometry

## 1 Introduction

We now study integrals involving powers of trigonometric functions. These integrals often do not have a direct antiderivative, but can be simplified using identities and substitution.

The key ideas are:

- use u-substitution when a derivative is present,
- use Pythagorean identities to rewrite remaining factors.

The identities we use most often are

$$\sin^2 x + \cos^2 x = 1, \quad 1 + \tan^2 x = \sec^2 x.$$

## 2 Products of $\sin x$ and $\cos x$

When integrating powers of  $\sin x$  and  $\cos x$ , we try to save one factor for  $du$  and convert the remaining powers using  $\sin^2 x = 1 - \cos^2 x$  or  $\cos^2 x = 1 - \sin^2 x$ .

### Example

Evaluate

$$\int \sin^3 x \cos^4 x \, dx.$$

Write

$$\sin^3 x = \sin x(1 - \cos^2 x).$$

Then

$$\int \sin^3 x \cos^4 x \, dx = \int \sin x(1 - \cos^2 x) \cos^4 x \, dx.$$

Let  $u = \cos x$ , so  $du = -\sin x \, dx$ . Then

$$= - \int (1 - u^2)u^4 \, du = - \int (u^4 - u^6) \, du.$$

$$= - \left( \frac{u^5}{5} - \frac{u^7}{7} \right) + C = -\frac{u^5}{5} + \frac{u^7}{7} + C.$$

Substitute back:

$$= -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C.$$

### Example

Evaluate

$$\int_0^{\pi/2} \sin^3 x \cos^2 x \, dx.$$

Write

$$\sin^3 x = \sin x(1 - \cos^2 x).$$

Let  $u = \cos x$ ,  $du = -\sin x \, dx$ .

$$x = 0 \Rightarrow u = 1, \quad x = \frac{\pi}{2} \Rightarrow u = 0.$$

Thus,

$$\begin{aligned} \int_0^{\pi/2} \sin^3 x \cos^2 x \, dx &= - \int_1^0 (1 - u^2)u^2 \, du = \int_0^1 (u^2 - u^4) \, du. \\ &= \left( \frac{u^3}{3} - \frac{u^5}{5} \right) \Big|_0^1 = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}. \end{aligned}$$

## 3 Products of $\sec x$ and $\tan x$

When working with  $\sec x$  and  $\tan x$ , we use

$$\frac{d}{dx}(\sec x) = \sec x \tan x, \quad \frac{d}{dx}(\tan x) = \sec^2 x,$$

and the identity

$$\sec^2 x = 1 + \tan^2 x.$$

### Example

Evaluate

$$\int \sec^4 x \tan^3 x \, dx.$$

Write

$$\tan^3 x = \tan x(\sec^2 x - 1).$$

Then

$$\int \sec^4 x \tan^3 x \, dx = \int \sec^4 x \tan x(\sec^2 x - 1) \, dx.$$

Let  $u = \sec x$ , so  $du = \sec x \tan x \, dx$ .

$$\begin{aligned} &= \int u^3(u^2 - 1) \, du = \int (u^5 - u^3) \, du. \\ &= \frac{u^6}{6} - \frac{u^4}{4} + C. \end{aligned}$$

Substitute back:

$$= \frac{\sec^6 x}{6} - \frac{\sec^4 x}{4} + C.$$

### Example

Evaluate

$$\int_0^{\pi/4} \sec^4 x \tan x \, dx.$$

Let  $u = \sec x$ ,  $du = \sec x \tan x \, dx$ .

$$x = 0 \Rightarrow u = 1, \quad x = \frac{\pi}{4} \Rightarrow u = \sqrt{2}.$$

Then

$$\int_0^{\pi/4} \sec^4 x \tan x \, dx = \int_1^{\sqrt{2}} u^3 \, du = \frac{u^4}{4} \Big|_1^{\sqrt{2}} = \frac{4}{4} - \frac{1}{4} = \frac{3}{4}.$$

## 4 Mixed Examples

### Example

Evaluate

$$\int \frac{\cos^3 x}{\sqrt{\sin x}} \, dx.$$

Write

$$\cos^3 x = \cos x(1 - \sin^2 x).$$

Let  $u = \sin x$ ,  $du = \cos x \, dx$ .

$$\begin{aligned} &= \int \frac{1 - u^2}{\sqrt{u}} \, du = \int (u^{-1/2} - u^{3/2}) \, du. \\ &= 2u^{1/2} - \frac{2}{5}u^{5/2} + C. \end{aligned}$$

Substitute back:

$$= 2\sqrt{\sin x} - \frac{2}{5}(\sin x)^{5/2} + C.$$

### Example

Evaluate

$$\int_0^{\pi/4} \frac{\tan^3 x}{\sqrt{\sec x}} \, dx.$$

Let  $u = \sec x$ ,  $du = \sec x \tan x \, dx$ , so  $\tan x \, dx = \frac{du}{u}$ .

$$x = 0 \Rightarrow u = 1, \quad x = \frac{\pi}{4} \Rightarrow u = \sqrt{2}.$$

$$\begin{aligned} \int_0^{\pi/4} \frac{\tan^3 x}{\sqrt{\sec x}} \, dx &= \int_1^{\sqrt{2}} \frac{u^2 - 1}{u^{3/2}} \, du = \int_1^{\sqrt{2}} (u^{1/2} - u^{-3/2}) \, du. \\ &= \left( \frac{2}{3}u^{3/2} + 2u^{-1/2} \right) \Big|_1^{\sqrt{2}}. \end{aligned}$$

## 5 Summary

Trigonometric integrals are evaluated using a combination of:

- u-substitution,
- Pythagorean identities,
- rewriting powers to expose derivatives.

The general strategy is:

1. save one factor for  $du$ ,
2. rewrite remaining powers using identities,
3. substitute and integrate.

For definite integrals, we always change bounds and evaluate using

$$F(x) \Big|_a^b.$$